INJECTIVITY PROPERTIES OF GENERALIZED WRONSKI MAPS

A Thesis

by

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ABSTRACT

Given a complex vector space V, consider the quotient map of the image of the Plücker embedding of the Grassmannian of m-planes of V by a certain subspace of $\mathbb{P} \wedge^m V$. Such maps generalize the classical Wronski maps on Grassmannians of spaces of polynomials, the Wronski maps on Grassmanians of spaces of solutions of linear homogeneous differential equations, pole-placement maps of input-output linear systems and their realizations as linear control systems. We are interested in finding the degree of such maps, i.e. in determining the number of points in the preimage of the generic point of the image. We distinguish a special subclass of these maps, called self-adjoint, for which the degree of the corresponding Wronski map is at least two. In the case of Wronski maps on Grassmanians of spaces of solutions of linear homogeneous differential equations our self-adjoint generalized Wronski maps correspond to the classical self-adjoint linear differential operators, up to a natural equivalence. In the case of linear control systems, they correspond to control system with symmetric transfer function, up to a state-feedback equivalence. The main question is whether there are non-selfadjoint generalized Wronski maps with the degree greater than 1. We give a negative answer to this question in the case $m = 2$ and $m = 3$ under some natural assumptions.

DEDICATION

I dedicate this thesis to my parents, Zhongyue Huang and Yali Zhang, for making me who I am, and my boyfriend Chuanhong Liu for loving and supporting me all the way.

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I would like to give my sincere thanks to my advisor Igor Zelenko, who gives me patient guidance in doing math research. During graduate studies, I learned a lot from him both in math and daily life. It is from him that I firstly learned what the phrase “pursuit of perfection” means. Before meeting him, I was not serious about doing math, and mostly I cared about “whether I can finish this”, but not “how well I can do this”. After this nearly two-year research with him, I realize that ’pursuit of perfection’ is an attitude towards life, and it is always better to try to do important things perfectly.

For most of the time, Dr. Zelenko is more like a friend. He has great passion for math, and is always willing to help when I am stuck in math or have some trouble in life. He cares about his students and is very patient answering their doubts. I will always bear his encouragement and advice in mind in later years’ exploring math.

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1. INTRODUCTION

1.1 Wronskian and Wronski map

Given a complex vector space V let $\mathrm{Gr}_m(V)$ be the Grassmannian of m-dimensional subspaces of V. In particular, $\mathrm{Gr}_1(V)$ is the projective space of V, written $(P)(V)$.

For simplicity, we will work with complex valued univariate C^∞ function on an interval I of $\mathbb{R}$, although the constructions make sense for functions which are smooth up to a sufficiently large order of the derivatives. Also, in the sequel, by a smooth function we will mean a C^∞ function. The *Wronskian* of m univariate C^∞ smooth functions $f_1(t), \dots f_m(t)$ on an interval I of $\mathbb{R}$ is the determinant

$$\mathrm{Wr}\big(f_1(t), f_2(t), \dots, f_m(t)\big) \;:=\; \det \begin{pmatrix} f_1(t) & f_2(t) & \dots & f_m(t) \\ f_1'(t) & f_2'(t) & \dots & f_m'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(m-1)}(t) & f_2^{(m-1)}(t) & \dots & f_m^{(m-1)}(t) \end{pmatrix}.$$

Given an m-dimensional subspace Λ in $C^\infty(I)$ choose a basis $\{f_i(t)\}_{i=1}^m$ of Λ and calculate its Wronskian $\mathrm{Wr}\big(f_1(t), f_2(t), \dots, f_m(t)\big)$. If we choose another basis $\{\tilde{f}_i(t)\}_{i=1}^m$ of Λ, then clearly

$$\mathrm{Wr}\big(\tilde{f}_1(t), \tilde{f}_2(t), \dots, \tilde{f}_m(t)\big) = c\mathrm{Wr}\big(f_1(t), f_2(t), \dots, f_m(t)\big),$$

where c is the determinant of the transition matrix between the bases. Therefore,

$$\{c\mathrm{Wr}\big(f_1(t), f_2(t), \dots, f_m(t)\big) : c \in \mathbb{C}\}$$

is the well-defines element of the set $\mathbb{P}C^\infty(I) \cup \{0\}$, where $\mathbb{P}C^\infty(I)$ denotes the projectivization of $C^\infty(I)$. This element is called the *Wronskian of the subspace* Λ and it is denoted by $\mathrm{Wr}(\Lambda)$.

In this way, one has the map Wr from the set of finite-dimensional subspaces of $C^\infty(I)$ to $\mathbb{P}C^\infty(I) \cup \{0\}$. Hence, given an n-dimensional subspace V in $C^\infty(I)$ and an integer $m < n$ the *Wronski map* $\mathrm{Wr}_{V,m}$ is the restriction of the map Wr to $\mathrm{Gr}_m(V)$.

Remark 1.1.1. Note that in general there are subspaces in the space of smooth functions with vanishing Wronskian (or equivalently there are linearly independent smooth functions with identically vanishing Wronskian), but for some classes of functions, for example, analytic functions, linear independence implies nonvanishing of Wronskian as was already noted by G. Peano in 1889, [10]).

Definition 1.1.2. We say that the map $F : X \to Y$,where Y is a topological space, is *strongly noninjective*, if the preimage of a generic point of $F(X)$ (in the relative topology) contains more than one point. We say that the map F is *essentially injective* if the preimage of a generic point of $F(X)$ (in the relative topology) contains exactly one point.

If X and Y are affine varieties over $\mathbb{C}$ and $F : X \to Y$ is a dominant finite rational map then the number of the preimages of a generic point is constant and is called the degree of the map F (see, for example, [8]). The map is strongly noninjective if its degree is greater than 1 and essentially injective if its degree is equal to 1. In particular, the map is ether strongly noninjective or essentially injective in this case.

Classically, in Algebraic Geometry, Wronski maps are considered in the case, where V is the space of polynomials (of degree $n-1$). Work of Schubert in 1886 [11], combined with a result of Eisenbud and Harris in 1983 [2] shows that in this case the Wronski map is surjective and the general polynomial in $\mathbb{P}\big(\mathrm{Pol}_{mp}(\mathbb{C})\big)$ has

$$\#_{m,p} \;:=\; \frac{1!2!\dots(p-1)!\cdot(mp)!}{m!(m+1)!\dots(m+p-1)!}$$

preimages, where $p = n - m$. Hence, it is strongly noninjective except for the trivial case $m = 1$ and $m = n - 1$.

In this paper we consider general spaces of smooth functions V, but we restrict ourselves to the case of $n = 2m$, i.e. the Grassmann of half-dimensional subspaces of an even dimensional space V.

The general question that we address here is:

Question 1 *Under what conditions on the space V the Wronski map* $\mathrm{Wr}_{V,m}$ *is strongly noninjective?*

1.2 Wronski map and linear differential operators

The point of departure for our study of the injectivity properties of Wronski map is the fact that the space of polynomials of degree not greater than $2m-1$ is the space of solutions of the differential equation $x^{(2m)} = 0$ of order $2m$, which is a formally self-adjoint differential equation. Consider a linear differential operator

$$Lx \;=\; x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \cdots + a_0(t)x(t) \tag{1.2.1}$$

of order n, whose coefficients a_0, a_1, $\dots$, a_{n-1} are complex-valued smooth functions on an interval I. Let V_L be the space of complex-valued solutions of the corresponding homogeneous differential equation $Lx = 0$.

Definition 1.2.1. The operator

$$L^*x \ := \ (-1)^n x^{(n)}(t) + \sum_{i=0}^{n-1}(-1)^i (a_i x)^{(i)} \tag{1.2.2}$$

is *(formally) adjoint* to operator L. The operator L (the differential equation $Lx = 0$) is *(formally) self-adjoint differential operator (equation)* if $L^* = L$.

If L is self-adjoint, then the order n of L must be even, $n = 2m$. Note that one can introduce the following natural equivalence relation on the set of linear differential operators:

Definition 1.2.2. We say that two linear differential operators L and $\widetilde{L}$ of the type (1.2.1) are equivalent if there exists a smooth function $\mu(\cdot)$ without zeros such that

$$\widetilde{L}x(\cdot) = \frac{1}{\mu(\cdot)} L\big(\mu(\cdot)x(\cdot)\big).$$

Remark 1.2.3. In each equivalence class of the linear operators with respect to the introduced equivalence relation there exists the unique operator of type (1.2.1) such that $a_{n-1}(t) \equiv 0$. This operator is called the *canonical representative* of the equivalence class. In particular, the canonical representative of the equivalence class of a self-adjoint operator is this self-adjoint operator itself.

The following theorem is a particular case of Theorem 1.5.2:

Theorem 1.2.4. *Assume that L is an arbitrary operator of order $2m$ equivalent to a self-adjoint operator. Then the Wronski map $\mathrm{Wr}_{V_L,m}$ is strongly non-injective.*

The proof of this theorem (or its more general version, Theorem 1.5.2 below) is based on the following two observations:

1. If L is equivalent to a self-adjoint operator then the space V_L is endowed with

a natural symplectic structure (see section 4.3)

2. If $\Lambda^{\angle}$ is a skew-orthogonal complement of an m-dimensional subspace Λ of V_L, then
$$\mathrm{Wr}(\Lambda^{\angle}) = \mathrm{Wr}(\Lambda), \tag{1.2.3}$$
i.e. the Wronskian is preserved under the skew-symmetric complement (the Lagrangian involution). This is proved (in more general setting) in section 2.

In the light of Theorem 1.2.4, the following question is natural:

Question 2 *Is there a linear differential operator L such that L is not equivalent to a self-adjoint operator and the Wronski map $\mathrm{Wr}_{V_L,m}$ is strongly non-injective?*

Now discuss under what condition an n-dimensional space V of smooth functions on an interval I is equal to V_L for some linear differential operator L of order n.

Proposition 1.2.5. *$V = V_L$ for some operator L as in* (1.2.1) *if and only if*
$$\mathrm{Wr}(V)(t) \neq 0, \quad \forall t \in I, \tag{1.2.4}$$

Proof. The necessity follows from the classical Abel Theorem. Conversely, assume that V satisfies (1.2.4). Take a basis $\phi_1(\cdot), \dots, \phi_n(\cdot)$ in V. Then from (1.2.4) it follows that the the vectors $\{\big(\phi_1^{(j)}(t), \dots, \phi_n^{(j)}(t)\big)\}_{j=0}^{n-1}$ form a basis of $\mathbb{C}^n$ for any $t \in I$. Therefore, there exist smooth functions $a_0(\cdot)$, $\dots a_{n-1}(\cdot)$ on I such that
$$\big(\phi_1^{(n)}(t), \dots, \phi_n^{(n)}(t)\big) = -\sum_{j=0}^{n-1} a_j(t)\big(\phi_1^{(j)}(t), \dots, \phi_n^{(j)}(t)\big), \tag{1.2.5}$$
which implies that if L is a linear differential operator as in (1.2.5) with coefficients a_j as in (1.2.5), then every function ϕ_j satisfies $Lx = 0$ and therefore $V = V_L$. $\square$

More generally, given an n-dimensional space V of smooth function on an interval I, there exist smooth functions $b_0(\cdot), \ldots b_{n-1}(\cdot)$ such that each function from V satisfies the differential equation

$$\mathrm{Wr}(V)(t)x^{(n)}(t) + b_{n-1}(t)x^{(n-1)}(t) + \ldots + b_0(t)x(t) = 0$$

The singular points of this differential equations are exactly zeros of the Wronskian $\mathrm{Wr}(V)$.

1.3 Generalized Wronski map

Trying to answer Questions 1 and 2, it is more convenient to use the following more general point of view. It turns out that the injectivity properties of the Wronski map $\mathrm{Wr}_{V,m}$ can be understood as the properties of intersections of the image of the Plücker embedding of $\mathrm{Gr}_m(V)$ with certain subspaces in $\mathbb{P} \wedge^m V$.

In more detail, given a subspace $\mathcal{R}$ of $\wedge^m V$, let $\hat{\pi}_R : \wedge^m V \to \wedge^m V/\mathcal{R}$ be the canonical projection. This induces a map $\pi_{\mathcal{R}} : \mathbb{P} \wedge^m V \to \mathbb{P} \wedge^m V/R \cup \{0\}$. Recall also that the Plücker embedding $\mathrm{Pl} : \mathrm{Gr}_m(V) \to \mathbb{P}\wedge^m V$ is the map defined as follows: if $\Lambda = span\{v_1, \ldots, v_m\} \in \mathrm{Gr}_m(V)$, then

$$\mathrm{Pl}(\Lambda) = [v_1 \wedge v_2 \wedge \ldots \wedge v_m],$$

where $[v_1 \wedge v_2 \wedge \ldots \wedge v_m]$ is the equivalence class of $v_1 \wedge v_2 \wedge \ldots \wedge v_m$ in $\mathbb{P} \wedge^m V$.

Definition 1.3.1. Two maps $F_1 : X_1 \to Y_1$ and $F_2 : X_2 \to Y_2$ are called equivalent if there are bijections $L : X_1 \to X_2$ and $R : Y_1 \to Y_2$ such that $R \circ F_1 = F_2 \circ L$.

As it is shown in Proposition 4.2.2 below, there exists a subspace $\mathcal{R}_V$ of $\wedge^m V$ such that the Wronski map $\mathrm{Wr}_{V,m}$ is equivalent to the map $\pi_{\mathcal{R}_V} \circ \mathrm{Pl}$ and therefore they have the same injectivity property. Since the Plücker embedding itself is injective this means that the injectivity property of $\mathrm{Wr}_{V,m}$ depends on the space $\mathcal{R}_V$ only.

Hence, the following question is a natural generalization of our Question 1:

Question 1′ *Let V be a vector space of dimension $2m$. Given a subspace $\mathcal{R}$ of $\wedge^m V$, consider the map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$. Under what condition on the space $\mathcal{R}$ this map is strongly noninjective?*

Question 1′ is more general than Question 1: the space V in Question 1′ is an abstract vector space and the map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ does not necessarily arise from the study of injectivity properties of the Wronski map for some space of functions. This motivates the following definition:

Definition 1.3.2. Given a subspace $\mathcal{R}$ of $\wedge^m V$ the map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is called a *generalized Wronski map* associated with $\mathcal{R}$. The subspace $\mathcal{R}$ is called the *defining subspace* of the generalized Wronski map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$.

1.4 Relation to pole placement problem for linear control systems

Generalized Wronski maps arise naturally in Control Theory, when one studies injectivity properties of the pole placement maps associated to a linear control system by a static output feedback. In more detail, given a triple $\Sigma = (A, B, C)$ of complex matrices of sizes $N \times N$, $N \times m$ $p \times N$ consider the following linear control system

$$\dot{x} = Ax + Bu, \tag{1.4.1}$$

$$y = Cx \tag{1.4.2}$$

where $x \in X = \mathbb{C}^N, y \in Y = \mathbb{C}^p, u \in U = \mathbb{C}^m$. Here X is called the *state space*, U is called the *input* (or *control*) *space*, and Y is called the *output space*. Choose an input (control) function $u(t)$ and an initial condition $x(0)$. Substituting $u(t)$ into (1.4.1), we obtain the trajectory $x(t)$ in the state space. Then substituting this trajectory into the state to output relation (1.4.2) we get also the trajectory $y(t)$.

We will assume that the system (1.4.1)-(1.4.2) is controllable and observable. Controllability means in general that any state in X can be reached by any other state at any positive time by applying a locally integrable control function $u(\cdot)$. For the linear system (1.4.1) it is equivalent to the fact the matrix $(B, AB, \dots A^{N-1}B)$ has rank N, where by the matrix $(B, AB, \dots A^{N-1}B)$ is obtained by attaching the columns of the matrices $B, AB, \dots A^{N-1}B$ one to each other. Observability means in general that from the knowledge of a control function $u(\cdot)$ and the output $y(\cdot)$ one can recover the state trajectory $x(\cdot)$ uniquely. For the linear system (1.4.1)-(1.4.2) this is equivalent to the fact that the matric $(C^T, A^T C^T, \dots (A^T)^{N-1} C^T)$ has rank N.

The assumption of controllability and observability means that N is equal to the *McMillan degree* of the transfer matrix valued function $G(s) = C(sI - A)^{-1}B$ of the system (1.4.1)-(1.4.2), obtained by applying the Laplace transform to it and using the initial conditions $x(0) = 0$, i.e. N is the minimal integer such that the function $G(s)$ is the transfer function of a control system of type (1.4.1)-(1.4.2).

Further, a static feedback is a linear map

$$u = Ky \tag{1.4.3}$$

from the output space to the input space which prescribes the dependence of the "future" input on the "past" output. Substituting (1.4.3) to (1.4.1) via (1.4.2) we obtain a linear homogeneous equation in the state space

$$\dot{x} = (A + BKC)x, \tag{1.4.4}$$

The *pole placement map* Pol_Σ associated with the control system (1.4.1)-(1.4.2) sends the $m\times p$ matrix K, defining the feedback transformation (1.4.3), to the characteristic polynomial $\det(\lambda I - A - BKC)$ of the matrix of the system (1.4.4). The name "pole placement" comes from the fact that the zeros of the polynomial $\mathrm{Pol}_\Sigma(K)$ (or, equivalently, the eigenvalues of the matrix of the equation (1.4.4)) are poles of the transfer matrix appearing in the formula for the Laplace transform of the equation (1.4.4)

Since K defines the linear map from Y to U it defines an element of $\mathrm{Gr}_p(Y\times U)$ being the graph of this linear map. Vice versa, any element of $\mathrm{Gr}_p(Y\times U)$ transversal to the subspace $0\times U$ is the graph of the linear map from Y to U and therefore defines a feedback of the form (1.4.3). Hence, the map Pol_Σ is well defined on the affine coordinate domain of $\mathrm{Gr}_p(Y\times U)$. Moreover, it can be extended to the map on the whole $\mathrm{Gr}_p(Y\times U)$, which is equivalent to a generalized Wronski map $\pi_{\mathcal{R}_\Sigma}\circ Pl$ for some subspace $\mathcal{R}_\Sigma$ of $\wedge^m(Y\times U)$.

In general, the pole placement problem consists of describing the image of the pole placement map. For example, if $N\le mp$ this image coincides with the space of polynomials of degree not greater than N ([13]) which means that any configuration of N eigenvalues (poles) of the matrix of (1.4.4) can be realized by a choice of an appropriate feedback (1.4.3).

In our situation $p=m$ and usually $N>mp$ so, the image of the pole placement map does not coincide with the space of polynomials of degree not greater than n. The Question $1'$ in this context of the pole placement map Pol_Σ asks under what conditions on the triple of matrices $\Sigma=(A,B,C)$ the corresponding control system (1.4.1)-(1.4.2) satisfies the following property: *among all polynomials of degree n (configurations of n points in the complex plane) that can be realized as characteristic polynomials (poles) of this control system after applying a feedback of type* (1.4.3), *a* *generic polynomial (configuration of points in $\mathbb{C}$) can be realized by more than one feedback* (1.4.3)?

The natural class of control systems that satisfies this property (and plays the same role among control systems as self-adjoint operators among linear differential operators) are symmetric control systems, i.e. control systems with

$$A = A^T, \quad C = B^T \tag{1.4.5}$$

Then in this case obviously $\mathrm{Pol}_\Sigma(K) = \mathrm{Pol}_\Sigma(K^T)$ for any $m \times m$ matrix K.

Furthermore, consider the following change of coordinates in the state, input, and output spaces

$$\begin{cases} x = L\widetilde{x} \\ u = Q\widetilde{y} + W\widetilde{u} \\ y = T\widetilde{y} \end{cases} \tag{1.4.6}$$

where L, W, and T are nonsingular matrices of sizes $N \times N$, $m \times m$, and $m \times m$ respectively, and Q is an $m \times m$ matrix, The transformation of the space $X \times U \times Y$ given by (1.4.6) is called a *state-feedback transformation* Substituting (1.4.6) into (1.4.1)-(1.4.2), we obtain a new linear control system in $\widetilde{x}$, $\widetilde{u}$, $\widetilde{y}$. We say that two linear control systems are *state-feedback* equivalent, if there is a state-feedback transformation transforming one system to another.

Obviously, the injectivity properties of the pole-placement map of a control system are preserved by a state-feedback transformation of this system. Therefore, *any control system which is state-feedback equivalent to a symmetric control system has strongly noninejctive pole placement map.* The question analogous to Question 2 is

Question 2′ *Are there controllable and observable linear control systems* (1.4.1)-(1.4.2) *with* $dimU = \dim Y$ *and with strongly noninjective pole placement map, which* *are not state-feedback equivalent to a symmetric control system?*

1.5 Self-adjoint generalized Wronski map

Trying to unify Question 2 and Question 2′, we have to extract the main common features of the Wronski map corresponding to a self-adjoint operator and of the pole placement map of a symmetric control system, which are responsible for their strong noninjectivity. Note that L is equivalent to a self-adjoint differential operator if and only if there exists a nondegenerate element $\sigma^* \in \wedge^2 V_L$ (i.e. a symplectic form on V_L^*) such that the space $\mathcal{R}_{V_L}$ contains the subspace of the form $\{\sigma^* \wedge \alpha : \alpha \in \wedge^{m-2} V_L\}$ (see Proposition 4.3.10). Here, as before, $\mathcal{R}_{V_L}$ is the subspace in $\wedge^m V_L$ such that the Wronski map $\mathrm{Wr}_{V_L,m}$ is equivalent to $\pi_{\mathcal{R}_{V_L}} \circ \mathrm{Pl}$. Similarly, a controllable and observable linear control system corresponding to the triple of matrices $\Sigma = (A, B, C)$ is state-feedback equivalent to a symmetric control system if and only if its pole placement map is equivalent to $\pi_{\mathcal{R}_\Sigma} \circ \mathrm{Pl}$, where $\mathcal{R}_\Sigma$ is a subspace of $\wedge^m(Y \times U)$ with a property that there exists a nondegenerate element $\sigma^* \in \wedge^2(Y \times U)$ (i.e. a symplectic form on $(Y \times U)^*$) such that the space $\mathcal{R}_\Sigma$ contains the subspace of the form $\{\sigma^* \wedge \alpha : \alpha \in \wedge^{m-2}(Y \times U)\}$ This motivates the following definition:

Definition 1.5.1. Given a $2m$ vector space V a subspace $\mathcal{R}$ of $\wedge^m V$ is called self-adjoint if there exists a symplectic form σ^* on V^* (considered as an element of $\wedge^2 V$) such that the space $\mathcal{R}$ contains the subspace of the form $\{\sigma^* \wedge \alpha : \alpha \in \wedge^{m-2} V\}$. In this case we also say that the generalized Wronski map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is self-adjoint.

Note that the symplectic form σ^* on V^* identifies V^* with V and therefore defines the symplectic form σ on V itself. Given a subspace Λ of V let $\Lambda^\angle$ be the skew-symmetric complement of Λ with respect to this form, i.e. $\Lambda^\angle = \{v \in V : \sigma(v, w) = 0 \quad \forall w \in \Lambda\}$. The following theorem is the generalization of Theorem 1.2.4:

Theorem 1.5.2. *Let V be a $2m$-dimensional vector space and $\mathcal{R}$ be a self-adjoint subspace of $\wedge^m V$. Then for any $\Lambda \in \mathrm{Gr}_m(V)$ we have*

$$\pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda^{\angle}) = \pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda). \tag{1.5.1}$$

In particular, the map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is strongly noninjective.

This theorem is proved in section 2.

The following assumption on the defining space $\mathcal{R}$ is natural, at least in the context of linear differential equations:

$$\mathbb{P}\mathcal{R} \cap \mathrm{Pl}\big(\mathrm{Gr}_m(V)\big) = \emptyset \tag{1.5.2}$$

Under this assumption the image of the map $\pi_R \circ \mathrm{Pl}$ belongs to $\mathbb{P}(\wedge^m V/\mathcal{R})$. By Remark 1.1.1, assumption (1.5.2) holds if $\mathcal{R} = \mathcal{R}_V$ and V is a subspace in the space of analytic functions. In particular, this will be the case if $V = V_L$ such that L is a linear differential operator with analytic coefficients.

In this more general setting the following question generalize Question 2 and $2'$(modulo assumption (1.5.2) in the case of control systems):

Question $2''$ *Given a $2m$ dimensional vector space V, is there a non-self-adjoint subspace $\mathcal{R}$ of $\wedge^m V$ such that it sutifies assumption* (1.5.2) *and the corresponding generalized Wronski map $\pi_R \circ \mathrm{Pl}$ is strongly non-injective?*

Proposition 1.5.3. *The answer to Question $2''$ is negative if $m = 2$.*

Proof. Indeed, assume that the generalized Wronski map $\mathcal{R}$ is strongly noninjective. Then $\mathcal{R} \neq 0$ (because the Plucker embedding is injective). If in addition, $\mathcal{R}$ satisfies (1.5.2), then there exist nondegenrate $\sigma^* \in \mathcal{R}$, which imples that $\mathcal{R}$ is self-adjoint. $\square$

Remark 1.5.4. As a matter of fact Proposition 1.5.3 is valid if assumption (1.5.2) is replaced by a weaker assumption

$$\mathbb{P}\mathcal{R} \setminus \mathrm{Pl}\big(\mathrm{Gr}_m(V)\big) \neq \emptyset \tag{1.5.3}$$

(in this case Proposition 1.5.3 is a tautology, because condition 1.5.3 is actually equivalent to self-adjointness of $\mathcal{R}$).

However, already for $m = 2$ without assuming (1.5.3) (which will imply that also (1.5.2) does not hold, if $\mathcal{R} \neq 0$) there are non-self adjoint $\mathcal{R}$ for which the generalized Wronski map is strongly noninjective. For example let W be a 3-dimensional subspace of 4-dimensional space V. Then the space $\mathcal{R} = \wedge^2 W \subset \wedge^2 V$ is not self-adjoint, because every element of $\wedge^2 W$ is decomposable, but the map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is strongly non-injective. Indeed, for any z_1 and z_2 in V with $z_1 \wedge z_2 \neq 0$ the plane generated by z_1 and z_2 intersects W non-trivially and if a nonzero x belong to this intersection then $z_1 \wedge z_2 + x \wedge w$ is decomposable so that the plane defined by this element and the plane $\mathrm{span}\{z_1, z_2\}$ belong to the same preimage of $\pi_{\mathcal{R}} \circ \mathrm{Pl}$.

Starting from $m = 3$ Question 2$''$ is more involved and we try to answer a simpler questions. Note that if $\mathcal{R}$ is self-adjoint then $\dim \mathcal{R} \geq \dim \wedge^{m-2} V = \binom{2m}{m-2}$,

Question 3 *Is there a non-self-adjoint subspace $\mathcal{R}$ of $\wedge^m V$ of dimension not greater than $\binom{2m}{m-2}$ and satisfying* (1.5.2) *such that the corresponding generalized Wronski map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is strongly non-injective?*

If the answer is no, then, in particular, it implies that if $\dim \mathcal{R} < \binom{2m}{m-2}$, then $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is essentially injective.

We prove that for $m = 3$ negative the answer to Question 3 is negative (see Corollary 3.3.2 and Theorem 3.3.4 below). The main idea of the proof is described in Remark (3.1.2). The consequences for the Wronski map associated with differential operators are given in Corollary 4.3.11 and for control system in Corollary 5.4.3.

Remark 1.5.5. Note that for $m = 2$ the answer to Question 3 is negative even without assumption (1.5.2), because in this case $\binom{2m}{m-2} = 1$ and if $\mathcal{R}$ is one dimensional and consists of decomposable elements generated by $x \wedge y$ with $x, y \in V$, then $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is not strongly noninjective, because for any z and w in V such that $x \wedge y \wedge z \wedge w \neq 0$ the preimage of $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ consist of the only one element $\mathrm{span}\{z, w\}$ of $\mathrm{Gr}_2(4)$.

2. SELF-ADJOINT GENERALIZED WRONSKI MAPS

In this section we give the proof of Theorem 1.5.2. Let $\mathcal{R}$ be a self-adjoint subspace of $\wedge^m V$ and let σ^* be a nondegenerate element of $\wedge^2 V$ (i.e. a symplectic form on V^*) such that the space $\{\sigma^* \wedge \alpha : \alpha \in \wedge^{m-2} V\}$ belongs to $\mathcal{R}$ and let σ be the induced symplectic form on V. Let Λ be an m-dimensional subspace in V. There is a Lagrangian subspace L_∞ of V such that $\Lambda \oplus L_\infty = V$, i.e such that Λ is transversal to L_∞.

Take another Lagrangian subspaces L_0 transversal to L_∞ in V. Then the sympletic form σ defines the natural identification between L_∞ and L_0^* via the map $v \in L_\infty \mapsto \sigma(v, \cdot) \in L_0^*$. Fix a basis $e_1, \dots, e_m$ on L_0 and take the dual basis $f_1, \dots, f_m$ of L_∞ ($\cong L_0^*$), i.e the basis satisfying $\sigma(f_i, e_j) = \delta_{ij}$. Recall that the constructed tuple $(e_1, \dots, e_m, f_1, \dots, f_m)$ is called a Darboux frame of the symplectic space V. By cosntructions the element σ^* of $\wedge^2 V$ satisfies

$$\sigma^* = \sum_{i=1}^{m} f_i \wedge e_i. \tag{2.0.1}$$

Any m-dimensional subspaces $\widetilde{\Lambda}$ of V which is transversal to L_∞ is the graph of the linear map $X_{\widetilde{\Lambda}} : L_0 \to L_\infty$, i.e. $\widetilde{\Lambda} = \{v + X_{\widetilde{\Lambda}} v : v \in L_0\}$. Let $S_{\widetilde{\Lambda}}$ be the matrix of the linear mapping $X_{\widetilde{\Lambda}}$ in the chosen bases $(e_1, \dots, e_m)$ and $(f_1, \dots, f_m)$ of L_0 and L_∞. The map $\widetilde{\Lambda} \mapsto S_{\widetilde{\Lambda}}$ defines a coordinate chart on the affine space $(\Lambda_\infty)^{\pitchfork}$ of all m-dimensional subspaces of V transversal to Λ_∞. In particular

Directly from definitions it follows that

$$S_{\widetilde{\Lambda}^\angle} = \left(S_{\widetilde{\Lambda}}\right)^T, \tag{2.0.2}$$

i.e the operation of taking skew-orthogonal complement corresponds to the matrix transposition in the corresponding coordinate chart.

Now assume that $S_\Lambda = (s_{ij})_{1\le i,j\le m}$. Then

$$\mathrm{Pl}(\Lambda) = \Big[\bigwedge_{i=1}^{m}(e_i + \sum_{j=1}^{m} s_{ji} f_j)\Big] \tag{2.0.3}$$

and by (2.0.2)

$$\mathrm{Pl}(\Lambda^{\angle}) = \Big[\bigwedge_{i=1}^{m}(e_i + \sum_{i=1}^{m} s_{ij} f_j)\Big]. \tag{2.0.4}$$

To prove the identity (1.5.1) it is sufficient to show that there exist $\alpha \in \wedge^{m-2}V$ such that

$$\bigwedge_{i=1}^{m}(e_i + \sum_{j=1}^{m} s_{ji} f_j) - \bigwedge_{i=1}^{m}(e_i + \sum_{j=1}^{m} s_{ij} f_j) = \sigma^* \wedge \alpha. \tag{2.0.5}$$

Given a multi-index $I = (i_1, \dots i_s)$ with $i_1 < \cdots < i_s$ denote $E_I = e_{i_1} \wedge \dots \wedge e_{i_s}$ and $F_I = f_{i_1} \wedge \dots \wedge f_{i_s}$. In the sequel $\#I$ denotes the number of indices in I and $|I|$ denotes the sum of indices in I, i.e. $\#I = s$ and $|I| = \sum_{k=1}^{s} i_k$. Also, let

$$\rho(I) = |I| - \frac{\#I(\#I+1)}{2}. \tag{2.0.6}$$

Given multi-indices I and J, denote by $(S_\Lambda)_{IJ}$ the submatrix of S_Λ obtained by the intersection of columns of S_Λ with indices from J and rows of S wih indices of I. Also by $\bar{I}$ denote the complement of I to $\{1,\dots,m\}$. Let I_m be the $m\times m$ identity matrix and $(I_m S_\Lambda)$ be the $m \times 2m$ matrix obtained by attaching the matrix S_Λ to I_m. Then it is easy to see that the coefficient of $E_I \wedge F_J$ in the decomposition of the right-hand side of (2.0.3) into the linear combination of $\Big\{E_{\widetilde{I}} \wedge F_{\widetilde{J}}\Big\}_{\#\widetilde{I}+\#\widetilde{J}=m}$ is equal to the

$(-1)^{\rho(I)}$ multiplied by $m \times m$ minor of the matrix $(I_m S_\Lambda)$ corresponding to columns appearing in I and $\underbrace{(m, m, \ldots, m)}_{\#J\text{times}} + J$, which in turn is equal to $(-1)^{\rho(I)} \det\Big((S_\Lambda)_{\bar{I}J}\Big)$. In other words,

$$\mathrm{Pl}(\Lambda) = \Big[\sum_{\{I,J:\#I+\#J=m\}} (-1)^{\rho(I)} \det\Big((S_\Lambda)_{\bar{I}J}\Big) E_I \wedge F_J\Big], \tag{2.0.7}$$

where all multiindices I and J in the last sum are strongly monotonic. This together with (2.0.4) implies that

$$\begin{aligned} &\bigwedge_{i=1}^{m}(e_i + \sum_{j=1}^{m} s_{ji} f_j) - \bigwedge_{i=1}^{m}(e_i + \sum_{j=1}^{m} s_{ij} f_j) = \\ &\sum_{\{I,J:\#I+\#J=m\}} (-1)^{\rho(I)} \Big(\det\Big((S_\Lambda)_{\bar{I}J} - (S_\Lambda^T)_{\bar{I}J}\Big) E_I \wedge F_J \end{aligned} \tag{2.0.8}$$

Now assume that

$$\alpha = \sum_{\{K,L:\#K+\#L=m-2\}} C_{K,L} E_K \wedge F_L, \tag{2.0.9}$$

where all multiindices K and L in the last sum are strongly monotonic.

Comparing coefficients in the right and left-hand sides of (2.0.5) with the help of (2.0.8) and (2.0.9), one gets a certain systems of linear equations for coefficients $C_{K,L}$ and we have to show that this system is consistent, i.e. has a solution.

First, if $I \cap J = \emptyset$ (or, equivalently, $I \cup J = \{1, \ldots, m\}$), then the diagonal of the submatrix $(S_\Lambda)_{\bar{I}J}$ is a subset of the diagonal of the matrix (S_Λ). Consequently, $\Big((S_\Lambda)_{\bar{I}J}\Big)^T = (S_\Lambda^T)_{\bar{I}J}$, which together with (2.0.8) implies that the left-hand side of (2.0.5) does not contain terms with $E_I \wedge F_J$ and $I \cap J = \emptyset$. Obviously, the right-hand side of (2.0.5) do not contain such terms as well for any $\alpha \in \wedge^{m-2} V$. In other words, coefficients of $E_I \wedge F_j$ with $I \cap J = \emptyset$ in both sides of (2.0.5) are equal to zero for any $\alpha \in \wedge^{m-2}V$.

In general, if $\#(I \cap J) = l+1$, let us suppose $M = I\backslash J$, $N = J\backslash I$. Note that in this case $M \cap N = \emptyset$. Opening the brackets in the expression $\sigma^* \wedge \alpha$, we can get a term containing $E_I \wedge F_J$ by wedging σ with terms containing $E_K \wedge F_L$ if and only if the sets K and L satisy the following propeties: $K \subset I$ and $L \subset J$, $\#K + \#L = m-2$ and $\#(K \cap L) = l$. In other wors K and L are obtained form I and J by removing one common element of I and J. Assume that

$$\overline{M \cup N} = (K \cap L) \cup \overline{(K \cup L)} = \{s_1, s_2, \ldots, s_{2l+2}\},$$

where $1 \leq s_1 < s_2 < \ldots < s_{2l+2} \leq m$. Given $U = \{u_1, \ldots, u_l\} \subset \{1, \ldots, 2l+2\}$, let $S_U = \{s_{u_1}, \ldots, s_{u_l}\}$ and

$$A_U := C_{M \cup S_U, N \cup S_U}. \tag{2.0.10}$$

Also, given $V = \{v_1, \ldots, v_{l+1}\} \subset \{1, \ldots, 2l+2\}$, let $S_V = \{s_{v_1}, \ldots s_{v_{l+1}}\}$ and

$$B_V = (-1)^{\rho(M \cup S_V)} \Big(\det\big((S_\Lambda)_{\overline{M \cup S_V}, N \cup S_V}\big) - \det\big((S_\Lambda^T)_{\overline{M \cup S_V}, N \cup S_V}\big)\Big). \tag{2.0.11}$$

Lemma 2.0.6. *Given $V = \{v_1, \ldots, v_{l+1}\} \subset \{1, \ldots, 2l+2\}$ let $\overline{V}$ be the complement of V to $\{1, \ldots, 2l+2\}$. Then*

$$B_{\overline{V}} = (-1)^{1+|\overline{M \cup N}|} B_V. \tag{2.0.12}$$

Proof. Note that by constructions $\overline{M \cup S_V} = N \cup S_{\overline{V}}$. Also, it is clear that

$$(S_\Lambda^T)_{IJ} = \Big((S_\Lambda)_{JI}\Big)^T.$$

Using this two facts, we get from the relation (2.0.11) that

$$B_V = (-1)^{\rho(M\cup S_V)}\Big(\det\big((S_\Lambda)_{N\cup S_{\overline{V}},N\cup S_V}\big) - \det\big((S_\Lambda)_{N\cup S_V,N\cup S_{\overline{V}}}\big)\Big). \tag{2.0.13}$$

Consequently,

$$B_{\overline{V}} = (-1)^{\rho(M\cup S_{\overline{V}})}\Big(\det\big((S_\Lambda)_{N\cup S_V,N\cup S_{\overline{V}}}\big) - \det\big((S_\Lambda)_{N\cup S_{\overline{V}},N\cup S_V}\big)\Big). \tag{2.0.14}$$

Therefore,

$$B_{\overline{V}} = -(-1)^{\rho(M\cup S_{\overline{V}})-\rho(M\cup S_V)}B_V. \tag{2.0.15}$$

Finally, from (2.0.6) it follows that

$$\rho(M\cup S_{\overline{V}}) - \rho(M\cup S_V) = |S_{\overline{V}}| - |S_V| \equiv |S_V| + |S_{\overline{V}}| \mod 2 \equiv |\overline{M\cup N}| \mod 2,$$

which together with (2.0.15) implies (2.0.12). □

Given $u \in \{1,\ldots,2l+2\}$ let

$$\nu(u) = s_u - u + \#M + l. \tag{2.0.16}$$

Note that given a set $U = \{u_1,\ldots,u_l\}$ the number $\nu(u)$ is exactly the number of the sign changes when one permutes $e_{s_u}\wedge f_{s_u}\wedge E_{M\cup S_U}\wedge F_{N\cup S_U}$ to the natural order of $e's$ and $f's$. Therefore for fixed M and N, we can get the following system of $\binom{2l+2}{l+1}$ linear equations with $\binom{2l+2}{l}$ unknown $A_{i_1,\ldots,i_l}$ from compariing coefficients in both sides of (2.0.5) near $E_I\wedge F_J$ with $M\subset I$ and $N\subset J$:

$$\sum_{c=1}^{l+1}(-1)^{\nu(u_c)}A_{u_1,\ldots,\widehat{u_c},\ldots,u_{l+1}} = B_{u_1,\ldots,u_{l+1}}, \quad \forall 1\le u_1<\ldots<u_{l+1}\le 2l+2. \tag{2.0.17}$$

Lemma 2.0.7. *The system* (2.0.17) *has a solution*

$$A_{u_1,\ldots,u_l} = \frac{(-1)^{\#M}}{(l+1)!} \sum_{2\le q_1<\ldots<q_l\le 2l+2} (-1)^{\mu(u_1,\ldots,u_l,q_1,\ldots,q_l)}\delta(u_1,\ldots,u_l,q_1,\ldots,q_l)B_{1,q_1,\ldots,q_l} \tag{2.0.18}$$

where

$$\mu(u_1,\ldots,u_l,q_1,\ldots,q_l) = t(u_1,\ldots,u_l,q_1,\ldots,q_l) + s_1 - 1 + \sum_{j=1}^{l}(s_{u_j} - u_j + s_{q_j} - q_j),$$

$$t(u_1,\ldots,u_l,q_1,\ldots,q_l) = \#(\{u_1,\ldots,u_l\}\cap\{1,q_1,\ldots,q_l\}),$$

$$\delta(u_1,\ldots,u_l,q_1,\ldots,q_l) = t!(l-t)! \text{ with } t = t(u_1,\ldots,u_l,q_1,\ldots,q_l).$$

Proof. It suffices to prove that (2.0.18) satisfies (2.0.17) for each fixed $1 \le u_1 < \ldots < u_{l+1} \le 2l+2$. Fix the set $\{q_1,\ldots,q_l\}$ and compute the coefficient of $B_{1,q_1,\ldots,q_l}$ after the substitution of the expressions for $A_{u_1,\ldots,\widehat{u_c},\ldots,u_{l+1}}$ from (2.0.18) into (2.0.17). Suppose that $p = \#(\{u_1,\ldots,u_{l+1}\}\cap\{1,q_1,\ldots,q_l\})$. Then there are two possibilities

1. $\#(\{u_1,\ldots,\widehat{u_c},\ldots,u_{l+1}\}\cap\{1,q_1,\ldots,q_l\}) = p-1$, which is equivalent to say $t(u_1,\ldots,\widehat{u_c},\ldots,u_{l+1},q_1,\ldots,q_l) = p-1$. Let

$$C_1 = \{c \in \{1,\ldots l+1\} : t(u_1,\ldots,\widehat{u_c},\ldots,u_{l+1},q_1,\ldots,q_l) = p-1\}.$$

   Note that $\#C_1 = p$.

2. $\#(\{u_1,\ldots,\widehat{u_c},\ldots,u_{l+1}\}\cap\{1,q_1,\ldots,q_l\}) = p$, which is equivalent to say that $t(u_1,\ldots,\widehat{u_c},\ldots,u_{l+1},q_1,\ldots,q_l) = p$. Let

$$C_2 = \{c \in \{1,\ldots l+1\} : t(u_1,\ldots,\widehat{u_c},\ldots,u_{l+1},q_1,\ldots,q_l) = p\}.$$

   Note that $\#C_2 = l+1-p$.

Note that by (2.0.16)

$$\nu(u_c)+\mu(\{u_1,\ldots,\widehat{u_c},\ldots,u_{l+1},q_1,\ldots,q_l)=$$
$$\begin{cases} \#M+l+\sum_{j=1}^{l+1}(s_{u_j}-u_j)+s_1-1+\sum_{j=1}^{l}(s_{q_j}-q_j)+p-1, & c\in C_1 \\ \#M+l+\sum_{j=1}^{l+1}(s_{u_j}-u_j)+s_1-1+\sum_{j=1}^{l}(s_{q_j}-q_j)+p, & c\in C_2. \end{cases}$$

There are three cases:

Case 1. Both sets C_1 and C_2 are not empty (or, equivalently, $0<p<l+1$) Then from the last relation it follows that the coefficient of $B_{1,q_1,\ldots,q_l}$ after the substitution of (2.0.18) to the left-hand side of (2.0.17) is equal to

$$\frac{(-1)^{l+\#M+\sum_{j=1}^{l+1}(s_{u_j}-u_j)+s_1-1+\sum_{j=1}^{l}(s_{q_j}-q_j)+p-1}}{(l+1)!}\big(p(p-1)!(l-(p-1))!-(l+1-p)(l-p)!p!\big)=0.$$

Case 2. C_2 is empty or, equivalently , $\{u_1,\ldots,u_{l+1}\}=\{1,q_1,\ldots,q_l\}$ or, equivalently, $p=l+1$. From (2.0.17) and (2.0.18) it follows that the coefficients of $B_{1,q_1,\ldots,q_l}=B_{u_1,\ldots,u_{l+1}}$ is equal to

$$\frac{1}{(l+1)!}l!(l+1)(-1)^{\text{ even number}}=1.$$

Case 3. C_1 is empty or, equivalently, $\{u_1,\ldots,u_{l+1}\}=\overline{\{1,q_1,\ldots,q_l\}}$ or, equivalently, $p=0$. Then the coefficient of $B_{1,q_1,\ldots,q_l}$ that the coefficient of $B_{1,q_1,\ldots,q_l}$ after the substitution of (2.0.18) to the left-hand side of (2.0.17) is equal to

$$\frac{(-1)^{\#M}l!(l+1)(-1)^{\#M+l+s_1-1+\ldots+s_{2l+2}-(2l+2)}}{(l+1)!}=$$
$$(-1)^{l+\sum_{j=1}^{2l+2}s_j-(2l+3)(l+1)}=(-1)^{1+|\overline{M\cup N}|}$$

(the latter equality follows from the fact that $l+(l+1)(2l+3)$ is always odd).

Combining cases 1 and 2 we get (2.0.17) in the case when $1 \in \{u_1, \dots u_{l+1}\}$. Combining cases 1 and 3 with the relation (2.0.12) from Lemma 2.0.6 we obtain (2.0.17) in the case when $1 \notin \{u_1, \dots u_{l+1}\}$.

The lemma is proved, thus completing the proof of Theorem 1.5.2. $\square$

3. GENERALIZED WRONSKI MAPS IN THE CASE OF $m = 3$

3.1 One-dimensional defining subspace: general observations

In this subsection we work with arbitrary m and vector space V of arbitrary dimension $n > m$, The following Lemma shows how noninjectivity properties of the generalized Wronski map with the defining subspace $\mathcal{R}$ is related to noninjectivity of the generalized Wronski map with the defining subspace $\mathcal{R}_1$ being a one-dimensional subspace of $\mathcal{R}$.

Lemma 3.1.1. *Given two distinct elements Λ_1 and Λ_2 of $\mathrm{Gr}_m(V)$ the identity $\pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_1) = \pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_2)$ holds if and only if there exist a line in $\mathcal{R}_1 \subset \mathcal{R}$ such that $\pi_{\mathcal{R}_1} \circ \mathrm{Pl}(\Lambda_1) = \pi_{\mathcal{R}_1} \circ \mathrm{Pl}(\Lambda_2)$.*

Proof. It is clear that if $\pi_{\mathcal{R}_1} \circ \mathrm{Pl}(\Lambda_1) = \pi_{\mathcal{R}_1} \circ \mathrm{Pl}(\Lambda_2)$ then for any subspace $\mathcal{R}$ containing $\mathcal{R}_1$ one has $\pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_1) = \pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_2)$.

In the opposite direction, if $\pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_1) = \pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_2)$, then consider the line $\widetilde{R}_1$ in $\mathbb{P} \wedge^m V$ connecting $\mathrm{Pl}(\Lambda_1)$ with $\mathrm{Pl}(\Lambda_2)$. Take the plane $\mathcal{R}_2$ in $\wedge^m V$ such that $\mathbb{P}\mathcal{R}_2 = \widetilde{R}_1$. Then any line $\mathcal{R}_1$ in the plane $\mathcal{R}_2$ such that $\mathbb{P}\mathcal{R}_1$ is different from $\mathrm{Pl}(\Lambda_1)$ and $\mathrm{Pl}(\Lambda_2)$ will serve our goal. □

If we denote by $\mathcal{S}_{\mathcal{R}}$ the following set

$$\mathcal{S}_{\mathcal{R}} = \{\Lambda \in \mathrm{Gr}_m(V) : \exists \widetilde{\Lambda} \neq \Lambda \text{ such that } \pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda) = \pi_{\mathcal{R}} \circ \mathrm{Pl}(\widetilde{\Lambda})\} \tag{3.1.1}$$

then the previous Lemma is equivalent to the following relation

$$\mathcal{S}_{\mathcal{R}} = \bigcup_{\mathcal{R}_1 \in \mathbb{P}R} S_{\mathcal{R}_1}. \tag{3.1.2}$$

Remark 3.1.2. Lemma (3.1.1) suggests the following scheme for study whether the map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is strongly noninjective: *first given any line $\mathcal{R}_1 \subset \mathcal{R}$ describe all pairs of elements in $\mathrm{Gr}_m(V)$ such that they belong to the same preimage of $\pi_{\mathcal{R}_1} \circ \mathrm{Pl}$. Then take a union of of all such pairs over all lines $\mathcal{R}_1$. If this union does not cover an open set of $\mathrm{Gr}_m(V)$ then the original map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is essentially injective. The group $\mathrm{GL}_n(V)$ acts naturally on $\mathrm{Gr}_m(V)$ and $\mathbb{P} \wedge^m V$ and obviously*

$$g.\big(\pi_{\mathcal{R}_1} \circ \mathrm{Pl}(\Lambda)\big) = \pi_{g.\mathcal{R}_1} \circ \mathrm{Pl}(g.\Lambda).$$

Therefore , in order to find the set of pairs in $\mathrm{Gr}_m(V)$ with the same image under $\pi_{\mathcal{R}_1} \circ \mathrm{Pl}$, it is enough to find it for one representative of the orbit of $\mathcal{R}_1$ with respect to the action of $\mathrm{GL}_n(V)$ on $\mathbb{P}\wedge^m V$. Usually this set is quite small (zero dimensional) for a generic line $\mathcal{R}_1$ in $\wedge^m V$. Therefore in order that the original map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ will be strongly noninjective, the defining subspace $\mathcal{R}$ must be of sufficiently big dimension or it must contain sufficiently big subset of degenerate lines. We implement this scheme for $m = 3$ and $n = 6$ in the next subsections.

Given any $\Lambda \in Gr_m(V)$, denote by $\Lambda^{\perp}$ the element of $\mathrm{Gr}_{n-m}(V^*)$ such that

$$\Lambda^{\perp} = \{p \in V^* : p(v) = 0 \quad \forall v \in \Lambda\} = \{p \in V^* : p|_{\Lambda} = 0\}. \tag{3.1.3}$$

The following proposition is useful in studying the case of $\dim \mathcal{R} = 1$:

Proposition 3.1.3. *Assume that $\mathcal{R}$ is one dimensional subspace of $\wedge^m V$, generated by ω. If Λ_1 and Λ_2 are two distinct m-dimensional subspaces of V such that $\pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_1) = \pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_2)$, then for any $1 \le k < m$, any k vectors $a_1, \dots, a_k$ in $\Lambda_1^{\perp}$, and*

$m-k$ vectors $b_{k+1},\dots b_{n-m}$ in $\Lambda_2^{\perp}$ we have

$$\omega(a_1,\dots,a_k,b_{k+1},\dots,b_m)=0, \tag{3.1.4}$$

where ω is considered as an m-form over V^*.

Proof. Let α_1 and α_2 be two elements of $\wedge^m V$ such that $\mathrm{Pl}(\Lambda_1)=[\alpha_1]$ and $\mathrm{Pl}(\Lambda_2)=[\alpha_2]$. Then directly from the definition

$$v_i \lrcorner \alpha_i = 0, \quad \forall v_i \in \Lambda_i^{\perp},\ i=1,2 \tag{3.1.5}$$

The condition $\pi_{\mathcal{R}}\circ\mathrm{Pl}(\Lambda_1)=\pi_{\mathcal{R}}\circ\mathrm{Pl}(\Lambda_2)$ with $\Lambda_1\neq\Lambda_2$ hold if and only if there exist nonzero constants b and c such that

$$\alpha_1 - b\alpha_2 = c\,\omega.$$

This together with (3.1.5) implies (3.1.4). □

As a direct consequence of Proposition 3.1.3 and Theorem 1.5.2 we get the following

Corollary 3.1.4. *If $m=2$, $\dim V=4$ and $\mathcal{R}$ is the one dimensional subspace of $\wedge^2 V$ generated by a nondegenerated element σ^* (i.e. a symplectic form on V^*), then Λ_1 and Λ_2 are two distinct 2-dimensional subspaces of V with $\pi_{\mathcal{R}}\circ\mathrm{Pl}(\Lambda_1)=\pi_{\mathcal{R}}\circ\mathrm{Pl}(\Lambda_2)$ if and only if $\Lambda_2=\Lambda_1^{\angle}$, where $\Lambda_1^{\angle}$ is the skew-symmetric complement of Λ_1 with respect to the symplectic form σ induced on V by the identification of V and V^* by the form σ^*. In particular, the degree of the map $\pi_{\mathcal{R}}\circ\mathrm{Pl}$ in this case is equal to 2.*

3.2 One-dimensional defining subspace: the case of $m = 3$ and $n = 6$

Assume that $\dim V = 6$. Following the scheme described in Remark (3.1.2) we need first to study the action of $\mathrm{GL}_6(V)$ on $\mathbb{P}\wedge^3 V$. The orbits under this action were described by Segre in 1918 ([12]). To formulate this result, let us introduce some notation: given a basis $\{e_i\}_{i=1}^6$ and the indices $i_1, i_2, \dots i_s$ let $e_{i_1,\dots i_s} = e_{i_1} \wedge \dots \wedge e_{i_s}$

Theorem 3.2.1. *(Segre [12], see also [1]) Assume that* $\dim V = 6$*. There are four orbits* $O_0, O_1, O_5,$ O_{10} *under the action of* $\mathrm{GL}_6(V)$ *on* $\mathbb{P}\wedge^3 V$*, where the lower index in* i *in* O_i *denotes the codimension of the orbit* O_i *in* $\mathbb{P}\wedge^3 V$*. Taking a basis* $\{e_i\}_{i=1}^6$ *in* V *representatives* ω_i *of orbits* O_i *are given by the following list:*

- $\omega_0 = e_{123} + e_{456}$
- $\omega_1 = e_{126} + e_{135} + e_{234}$
- $\omega_5 = e_1 \wedge (e_{23} + e_{45})$
- $\omega_{10} = e_{123}$

Remark 3.2.2. Recall that the tangential variety $\mathcal{T}X$ of a projective variety X in a projective space $\mathbb{P}^N$ is the union of all tangent lines X. The orbits of Theorem 3.2.1 can be also describe geometrically as follows:

- The orbit O_0 is the complement of the tangential variety $\mathcal{T}\mathrm{Pl}\big(\mathrm{Gr}_3(V)\big)$ of $\mathrm{Pl}\Big(\mathrm{Gr}_3(V)\Big)$ to $\mathbb{P}\wedge^m V$;
- To describe the orbit O_1 let $\mathcal{T}_1$ be the union of all lines in $\mathbb{P}\wedge^m V$ connecting two points in $\mathrm{Pl}\Big((\mathrm{Gr}_3(V)\Big)$, corresponding to two 3-dimensional subspace in V having nonzero intersection. Then

$$\mathrm{Pl}\Big(Gr_3(V)\Big) \subset \mathcal{T}_1 \subset \mathcal{T}\mathrm{Pl}\big(\mathrm{Gr}_3(V)\big)$$

and O_1 is the complement of $\mathcal{T}_1$ to $\mathcal{T}\mathrm{Pl}\big(\mathrm{Gr}_3(V)\big)$;

- The orbit O_5 is the complement of $\mathrm{Pl}\big(\mathrm{Gr}_3(V)\big)$ to $\mathcal{T}_1$;
- the orbit O_{10} coincedes with $\mathrm{Pl}\big(\mathrm{Gr}_3(V)\big)$.

Having the description of orbits given by Theorem 3.2.1, we can examine the injectivity properties of the generalized Wronski map corresponding to a representative of each orbit as a defining subspace. The Proposition 3.1.3 is very useful for this goal.

Proposition 3.2.3. *Assume that* $\dim V = 6$*,* $\mathcal{R}$ *is a one-dimensional subspace of* $\wedge^3 V$*,* $\mathcal{R} \not\subset \mathrm{Pl}(\mathrm{Gr}_3(V))$*, and* ω *be a generator of* $\mathcal{R}$*. Then the condition* $\pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_1) = \pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_2)$ *with* $\Lambda_1 \neq \Lambda_2$ *implies that* $\dim(\Lambda_1 \cap \Lambda_2) \leq 1$ *and for any* $a \in \Lambda_1^{\perp} \cup \Lambda_2^{\perp}$ *we have that* $a \lrcorner\, \omega$ *has rank not greater than* 2*.*

Proof. Let $W = \Lambda_1^{\perp} \cup \Lambda_2^{\perp}$ and $W_1 = \operatorname{span} W$. Let us consider several cases separately:

Case 1. $\dim W_1 = 6$. For any $a \in \Lambda_1^{\perp}$, by using Proposition 3.1.3 we have $\Lambda_2^{\perp} \cup a \subset \ker(a \lrcorner\, \omega)$. Since dim span$(\Lambda_2^{\perp} \cup a) = 4$, the rank of $a \lrcorner\, \omega$ is not bigger than 2. This is also true for all $a \in \Lambda_2^{\perp}$. Therefore the above claim holds for all $a \in W$,

Case 2. $\dim W_1 = 5$. If $a \in \Lambda_1^{\perp} \backslash \Lambda_2^{\perp}$, then by the same argument as above, we have $\dim \ker(a \lrcorner\, \omega)|_{W_1} \geq 4$, so $\dim \ker(a \lrcorner\, \omega) \geq 3$. Since the kernel should be of even dimension, we have $\dim \ker(a \lrcorner\, \omega) \geq 4$, which means $\operatorname{rank}(a \lrcorner\, \omega) \leq 2$. The same arguments hold for the case when $a \in \Lambda_2^{\perp}/\Lambda_1^{\perp}$. If $a \in \Lambda_1^{\perp} \cap \Lambda_2^{\perp}$, then $\dim \ker(a \lrcorner\, \omega)|_{W_1} \geq 5$, so $\dim \ker(a \lrcorner\, \omega)|_V \geq 4$, which means $\operatorname{rank}(a \lrcorner\, \omega) \leq 2$.

Case 3. $\dim W_1 = 4$. It means $\dim(\Lambda_1^{\perp} \cap \Lambda_2^{\perp}) = 2$. It is easy to show that by group action of $\mathrm{GL}_6(V)$, ω can be brought to the normal form e_{123}. It means that $\omega \in \mathrm{Pl}(\mathrm{Gr}_3(V))$ in contradiction to our assumptions.

Finally, note that $\dim(\Lambda_1 \cap \Lambda_2) \leq 1$, because otherwise either $\Lambda_1 = \Lambda_2$ or $\mathrm{Pl}(\Lambda_1) - \mathrm{Pl}(\Lambda_2)$ belongs to $\mathbb{P}\mathcal{R} \cap \mathrm{Pl}\big(\mathrm{Gr}_3(V)\big)$.

$\square$

Using the last proposition, let us check what will happen if ω lies in different orbits.

Proposition 3.2.4. *Assume that* $\dim V = 6$*, Let* $\mathcal{R}$ *be a one-dimensional subspace of* $\wedge^3 V$ *such that* $\mathbb{P}\mathcal{R} \notin \mathrm{Pl}(\mathrm{Gr}_3(V))$*. Let* $\mathcal{S}_{\mathcal{R}}$ *be as in* (3.1.1)*. Then*

1. *If* $\mathbb{P}\mathcal{R} \in O_0$*, then there is only one pair of distinct* 3*-planes with the same image under* $\pi_{\mathcal{R}} \circ \mathrm{Pl}$*. In particular* $\dim \mathcal{S}_{\mathcal{R}} = 0$
2. *If* $\mathbb{P}\mathcal{R} \in O_1$*, then* $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ *is classically injective, i.e.* $\mathcal{S}_{\mathcal{R}} = \emptyset$
3. *If* $\mathbb{P}\mathcal{R} \in O_5$*, then* $\dim S_{\mathcal{R}} = 4$ *i.e. the family of pairs of distinct* 3*-planes with the same image under* $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ *is a four dimensional manifold.*

Proof. Let $\{dx_i\}_{i=1}^6$ be the basis of V^* dual to $\{e_i\}_{i=1}^6$ in V and ω_i be as in Theorem 3.2.1. Let

$$H_\omega = \{a \in V^* | \mathrm{rank}(a \lrcorner \omega) \leq 2\}$$

Proposition 3.2.3 implies that if $\pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_1) = \pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_2)$ then

$$\Lambda_1^\perp \cup \Lambda_2^\perp \subset H_\omega \tag{3.2.1}$$

By direct calculations

$$H_{\omega_0} = \mathrm{span}\{dx_1, dx_2, dx_3\} \cup \mathrm{span}\{dx_4, dx_5, dx_6\} \tag{3.2.2}$$

$$H_{\omega_1} = \mathrm{span}\{dx_4, dx_5, dx_6\} \tag{3.2.3}$$

$$H_{\omega_5} = \mathrm{span}\{dx_2, dx_3, dx_4, dx_5, dx_6\} \tag{3.2.4}$$

1. If $\omega = \omega_0$, then from (3.2.1) and (3.2.2) it follows that the only pair of 3-planes with the same image of $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is $\Big((\mathrm{span}\{dx_1, dx_2, dx_3\})^{\perp}, (\mathrm{span}\{dx_4, dx_5, dx_6\})^{\perp}\Big)$, which proves item 1 of the proposition.

2. If $\omega = \omega_1$, then from (3.2.1) and (3.2.2) it follows that there are no distinct 3-planes with the same image of $\pi_{\mathcal{R}} \circ \mathrm{Pl}$, which proves item (2) of the proposition.

3. If $\omega = \omega_5$, then from (3.2.1) and (3.2.4) it follows that

$$\Lambda_1 \cap \Lambda_2 \supset H_{\omega_5}^{\perp} = \mathrm{span}\,\{e_1\}. \tag{3.2.5}$$

Besides, since $\dim H_{\omega_5} = 5$, Proposition 3.2.3 implies that $\dim(\Lambda_1^{\perp} \cap \Lambda_2^{\perp}) = 1$. From the arguments of case 2 of Proposition 3.2.3 it follows that if $a \in \Lambda_1^{\perp} \cap \Lambda_2^{\perp}$ then $\dim \ker(a \lrcorner\, \omega)|_{H_{\omega_5}} \geq 5$. From the form of ω_5 it follows directly that $a \in \mathrm{span}\,\{dx_6\}$. Therefore

$$\mathrm{span}(\Lambda_1 \cup \Lambda_2) = \mathrm{span}\,\{e_1, e_2, e_3, e_4, e_5\}. \tag{3.2.6}$$

From (3.2.5) and (3.2.6) it follows that there exist $\sigma_1, \sigma_2 \in \wedge^2 \mathrm{span}(\Lambda_1 \cup \Lambda_2)$ such that

$$\mathrm{Pl}(\Lambda_1) = e_1 \wedge \sigma_1, \quad \mathrm{Pl}(\Lambda_2) = e_1 \wedge \sigma_2 \tag{3.2.7}$$

Let $\sigma = e_{23} + e_{45}$, then $\omega_5 = e_1 \wedge \sigma$. Recall that the condition $\pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_1) = \pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_2)$ is equivalent to the fact that there exists real $c \neq 0$ such that $\mathrm{Pl}(\Lambda_1) - \mathrm{Pl}(\Lambda_2) = c\omega_5$. Then from (3.2.7) it follows that

$$e_1 \wedge (\sigma_1 - \sigma_2) = c e_1 \wedge \sigma,$$

which in turn implies that

$$\sigma_1 - \sigma_2 = c\sigma \ \mathrm{mod} e_1 \tag{3.2.8}$$

Let $\widehat{V} = \mathrm{span}(\Lambda_1 \cup \Lambda_2)/\mathrm{span}\{e_1\}$. Note that $\dim \widehat{V} = 4$. Let $\mathrm{Pr} : \mathrm{span}(\Lambda_1 \cup \Lambda_2) \to \widehat{V}$ be the canonical projection. Let $\widehat{\mathrm{Pl}}$ be the Plücker embedding of $\mathrm{Gr}_2(\widehat{V})$

Then $\sigma_1 = \widehat{\mathrm{Pl}}\big(\mathrm{Pr}(\Lambda_1)\big)$ and $\sigma_2 = \widehat{\mathrm{Pl}}\big(\mathrm{Pr}(\Lambda_2)\big)$ and (3.2.8) is equivalent to the following identity

$$\widehat{\mathrm{Pl}}\big(\mathrm{Pr}(\Lambda_1)\big) - \widehat{\mathrm{Pl}}\big(\mathrm{Pr}(\Lambda_2)\big) = c\widehat{\sigma}. \tag{3.2.9}$$

where $\widehat{\sigma}$ is the pushforward of σ to $\widehat{V}$. Note that $\widehat{\sigma}$ is nondegenerate.

In other words, if $\widehat{\mathcal{R}}$ denotes a subspace in $\wedge^2\widehat{V}$ generated by σ then

$$\pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_1) = \pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_2) \Leftrightarrow \pi_{\widehat{\mathcal{R}}} \circ \widehat{\mathrm{Pl}}\big(\mathrm{Pr}(\Lambda_1)\big) = \pi_{\widehat{\mathcal{R}}} \circ \widehat{\mathrm{Pl}}\big(\mathrm{Pr}(\Lambda_2)\big).$$

From this and Corollary 3.1.4 it follows that

$$\pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_1) = \pi_{\mathcal{R}} \circ \mathrm{Pl}(\Lambda_2) \Leftrightarrow \mathrm{Pr}(\Lambda_2) = \Big(\mathrm{Pr}(\Lambda_1)\Big)^{\angle} \text{ with respect to } \widehat{\sigma}.$$

Therefore the set of pair of distinct 3-planes with same image of $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is parametrized by the set of non-Lagrangian 2-planes in $\widehat{V}$, which is 4 dimensional. □

3.3 The case of defining subspace of dimension less than 6

From Proposition 3.2.4 and (3.1.2) it follows that under the assumption (1.5.2)

$$\mathcal{S}_{\mathcal{R}} = \bigcup_{\mathcal{R}_1 \in \mathbb{P}\mathcal{R} \cap O_0} S_{\mathcal{R}_1} \cup \bigcup_{\mathcal{R}_1 \in \mathbb{P}\mathcal{R} \cap O_5} S_{\mathcal{R}_1} \tag{3.3.1}$$

Then from items 1 and 3 of Proposition 3.2.4 it follows that

$$\dim S_{\mathcal{R}} \leq \max\{\dim(PR \cap O_0), \dim(\mathbb{P}R \cap O_5) + 4\} \tag{3.3.2}$$

Since $\dim Gr_3(V) = 9$, the last relation implies the following

Theorem 3.3.1. *If $\mathcal{R}$ satisfies* (1.5.2), $\dim \mathcal{R} < 10$ *and* $\dim \mathbb{P}\mathcal{R} \cap O_5 \leq 4$*, then the generalized Wronski map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is essentially injective.*

Proof. Indeed , from the assumptions and (3.3.2) it follows that $\dim S_{\mathcal{R}} \leq 8$. Therefore $Gr_3(V) \backslash S_{\mathcal{R}}$ is a nonempty Zariski open set and therefore the map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is essentially injective. □

Corollary 3.3.2. *If $\mathcal{R}$ satisfies* (1.5.2) *and* $\dim \mathcal{R} \leq 5$*, then the generalized Wronski map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is essentially injective.*

3.3.1 The case of 6-dimensional defining subspace

In this subsection we consider the case when the following three conditions hold

1. $\dim \mathcal{R} = 6$;
2. $\dim \mathbb{P}\mathcal{R} \cap O_5 \geq 5$, which together with the first condition is equivalent to $\dim \mathbb{P}\mathcal{R} \cap O_5 = 5$;
3. $\mathcal{R}$ satisfies assumption (1.5.2).

If $\mathcal{R}$ is self-adjoint with $\dim \mathcal{R} = 6$, then $\mathcal{R} \subset O_5$ and therefore $\mathcal{R}$ satisfies all three condition above. Our goal is to show the following

Theorem 3.3.3. *Six dimensional self-adjoint subspaces are the only subspaces of $\wedge^3 V$ satisfying conditions (1)-(3) above.*

As a direct consequence of this theorem and Theorem 1.5.2 and Corollary 3.3.2 we get

Theorem 3.3.4. *If $\mathcal{R}$ satisfies* (1.5.2), $\dim \mathcal{R} = 6$*, and the generalized Wronski map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is strongly non-injective, then $\mathcal{R}$ is self-adjoint.*

Proof of Theorem 3.3.3

First from conditions (1) and (2) it follows that $\mathbb{P}\mathcal{R} \subset \overline{O_5} = O_5 \cup O_{10}$. Then from condition (3) we obtain that

$$\mathbb{P}\mathcal{R} \subset O_5 \tag{3.3.3}$$

The following lemma gives the characterization of lines in O_5 based on the classical Weierstrass and Kronecker theory of pencils of skew-symmetric forms [5, 6].

Lemma 3.3.5. *Assume that $\omega_1, \omega_1 \in O_5$. i.e. $\omega_1 = \alpha_1 \wedge \sigma_1$ and $\omega_2 = \alpha_2 \wedge \sigma_2$ for some nondegenerate σ_1 and σ_2 in $\wedge^2 V$. Then*

$$\lambda\omega_1 + \mu\omega_2 \in O_5 \ \forall \lambda, \mu \tag{3.3.4}$$

if and only if one of the following two conditions holds:

1. $\sigma_1 = c\sigma_2 \mod (\alpha_1, \alpha_2)$;

2. $\alpha_1 = c\alpha_2$.

Proof. It is clear that if one of the two cases hold then $\lambda\omega_1 + \mu\omega_2 \in O_5 \ \forall \lambda, \mu$ holds.

In opposite direction it suffices to prove that if

$$\alpha_1 \neq c\alpha_2, \tag{3.3.5}$$

then

$$\sigma_1 = c\sigma_2 \mod (\alpha_1, \alpha_2). \tag{3.3.6}$$

The latter is equivalent to the following: if W denotes the four dimensional subspace of V^* which annihilates both α_1 and α_2, then

$$\sigma_1|_W = \sigma_2|_W \tag{3.3.7}$$

First let us bring the pair $\alpha_1 \wedge \sigma_1$, $\alpha_2 \wedge \sigma_2$ to some convenient form by action of $\mathrm{GL}_6(V)$ or, equivalently, by choosing an appropriate basis $\{e_i\}_{i=1}^6$ in V. First, by assumption (3.3.5) we can make $e_1 = \alpha_1$ and $e_2 = \alpha_2$. Now consider the pair of symplectic forms $(\sigma_1|_W, \sigma_2|_W)$ on W. The pencil of skew-symmetric forms $\lambda\sigma_1|_W + \mu\sigma_2|_W$ generated by these forms is regular and according to the classical Weierstrass theorem is determined by its elementary divisors (see [5], [6]). Since $\dim M = 4$ and each elementary divisor of the skew-symmetric pencils appear twice, there are essentially two cases here:

(a) Two linear elementary divisors, which appear twice;

(b) One elementary divisor which is a square of a linear form, which appear twice.

Then, maybe after multiplying σ_2 by a non-zero constant if necessary, one can introduce a basis $\{\tilde{e}_i\}_{i=3}^6$ in $W^* = V/\mathrm{span}\{e_1, e_2\}$ such that

$$\sigma_1|_W = \widetilde{e}_{34} + \widetilde{e}_{56} \tag{3.3.8}$$

and

$$\sigma_2|_W = \widetilde{e}_{34} + a\widetilde{e}_{56} \tag{3.3.9}$$

in Case (a) or

$$\sigma_2|_W = a\widetilde{e}_{34} + a\widetilde{e}_{56} + \widetilde{e}_{45} \tag{3.3.10}$$

in Case (b) , where a is some nonzero constant and $\widetilde{e}_{ij}$ stands for $\tilde{e}_i \wedge \tilde{e}_j$. Completing the previously chosen pair (e_1, e_2) to the basis $\{e_i\}_{i=1}^6$ such that $e_i|_W = \tilde{e}_i$ for $3 \leq i \leq 6$ we get that

$$\alpha_1 \wedge \sigma_1 = e_1 \wedge \big((e_{34} + e_{56}) + e_2 \wedge (p_{13}e_3 + p_{14}e_4 + p_{15}e_5 + p_{16}e_6)\big) \tag{3.3.11}$$

and

$$\alpha_2 \wedge \sigma_2 = e_2 \wedge \big((e_{34} + a e_{56} + e_1 \wedge (p_{23} e_3 + p_{24} e_4 + p_{25} e_5 + p_{26} e_6) \big) \tag{3.3.12}$$

in Case (a) or

$$\alpha_2 \wedge \sigma_2 = e_2 \wedge \big((a e_{34} + a e_{56}) + e_{45} + e_1 \wedge (p_{23} e_3 + p_{24} e_4 + p_{25} e_5 + p_{26} e_6) \big) \tag{3.3.13}$$

in Case (b), where $p_{13}, p_{14}, p_{15}, p_{16}, p_{23}, p_{24}, p_{25}, p_{26}$ are constant coefficients.

Note that by constructions we have the freedom to make the changes of basis of the type $e_i \mapsto e_i + a_i e_1 + b_i e_2$ for all i, $3 \leq i \leq 6$. Then by choosing an appropriate constants a_i and b_i we can make all coefficient p_{ij} in all equations (3.3.11)-(3.3.13) equal to zero, i.e. we can obtain the following normal forms

$$\alpha_1 \wedge \sigma_1 = e_1 \wedge \big(e_{34} + e_{56} \big) \tag{3.3.14}$$

and

$$\alpha_2 \wedge \sigma_2 = e_2 \wedge \big(e_{34} + a e_{56} \big) \tag{3.3.15}$$

in Case (a) or

$$\alpha_2 \wedge \sigma_2 = e_2 \wedge \big((a e_{34} + a e_{56}) + e_{45}) \big) \tag{3.3.16}$$

in Case (b).

Note that directly from normal forms in Theorem 3.2.1 it follows that $\overline{O_5}$ is characterized by the following property: for any $\omega \in \overline{O}_5$, there exists $v \in V^*$ such that v annihilates ω. Assuming that (3.3.5) holds let us see under what condition such v exists for $\lambda \alpha_1 \wedge \sigma_1 + \mu \alpha_2 \wedge \sigma_2$ for every λ and μ. Suppose that $v = \sum_{i=1}^{6} v_i dx_i$, where $\{dx_i\}_{i=1}^{6}$ is the dual basis to $\{e_i\}_{i=1}^{6}$.

Consider Cases (a) and (b) separately:

1. For Case (a)

$$\begin{aligned}&v \lrcorner\, (\lambda\alpha_1 \wedge \sigma_1 + \mu\alpha_2 \wedge \sigma_2) = (\lambda v_1 + \mu v_2)e_{34} + (\lambda v_1 + a\mu v_2)e_{56} - \lambda v_3 e_{14} + \\ &\lambda v_4 e_{13} - \lambda v_5 e_{16} + \lambda v_6 e_{15} - a\mu v_3 e_{24} + \mu v_4 e_{23} - \mu a v_5 e_{26} + \mu a v_6 e_{25} = 0\end{aligned} \tag{3.3.17}$$

which has nontrivial solution if and only if $a = 1$. Recall that during normalization we allowed to multiply σ_2 by a non-zero constant, therefore $a = 1$ implies that the original σ_1 and σ_2 satisfy (3.3.6).

2. For the Case (b), similarly

$$\begin{aligned}&v \lrcorner\, (\lambda\alpha_1 \wedge \sigma_1 + \mu\alpha_2 \wedge \sigma_2) = (\lambda v_1 + \mu a v_2)e_{34} - \lambda v_3 e_{14} + \\ &\lambda v_4 e_{13} + (\lambda v_1 + \mu a v_2)e_{56} - \lambda v_5 e_{16} + \lambda v_6 e_{15} + \mu(-a v_3 + v_5)e_{24} \\ &+ \mu a v_4 e_{23} - \mu a v_5 e_{26} + \mu(a v_6 - v_4)e_{25} + \mu v_2 e_{45} = 0,\end{aligned} \tag{3.3.18}$$

which implies that $v = 0$, if both λ and μ are not zero. This show that in this case (3.3.4) is impossible. $\square$

Remark 3.3.6. If $\sigma_1 = \sigma_2 \mod (\alpha_1, \alpha_2)$, we can find σ such that

$$\alpha_1 \wedge \sigma_1 = \alpha_1 \wedge \sigma, \quad \alpha_2 \wedge \sigma_2 = \alpha_2 \wedge \sigma, \tag{3.3.19}$$

Indeed, there exist β_1, β_2 in V such that

$$\sigma = \sigma_1 + \alpha_1 \wedge \beta_1 = \sigma_2 + \alpha_2 \wedge \beta_2,$$

which implies (3.3.6).

From (3.2.4) it follows that if $\omega \in O_5$, i.e. $\omega = \alpha \wedge \sigma$, where $\alpha \in V$ and σ is a non-decomposible element of $\wedge^2 V$, then α is defined uniquely up to a nonzero constant. Now fix a basis $\{\alpha_i \wedge \sigma_i\}_{i=1}^6$ of $\mathcal{R}$ and let $L = \operatorname{span}\{\alpha_i\}_{i=1}^6$.

Lemma 3.3.7. *If $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is strongly non-injective, then $\dim L \geq 4$.*

Proof. Let $j = \dim L$, $K = L^{\perp}$, and $\mathcal{S}_{\mathcal{R}}$ be as in (3.1.1). If $\Lambda_1 \in S_{\mathcal{R}}$, then by (3.2.4) $\Lambda_1^{\perp}$ is annihilated by some nonzero element of L.

Assume by contradiction that $j \leq 3$. Then the set of all Λ_1 such that $\Lambda_1^{\perp}$ is transversal to K in V^* is open in Zariski topology. For all such Λ_1, if an element of L annihilates $\Lambda_1^{\perp}$ then it annihilates the whole V^*, namely it is zero element. Therefore all such Λ_1 do not belong to $\mathcal{S}_R$ and the map $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is essentially injective. Hence, $j \geq 4$. $\square$

Lemma 3.3.8. *Given $k+1$ linearly idependent elements $\{\alpha_i\}_{i=1}^{k+1}$ in V, if $\omega \in \wedge^2 V$ satisfies*

$$\omega = 0 \mod (\alpha_i, \alpha_{k+1}), \quad \forall i \in \{1, \ldots, k\}, \tag{3.3.20}$$

then

$$\omega = \begin{cases} c\alpha_1 \wedge \alpha_2 \mod \alpha_{k+1}, & k = 2 \\ 0 \qquad\qquad \mod \alpha_{k+1}, & k > 2, \end{cases} \tag{3.3.21}$$

where c is some constant in the case $k = 2$.

Proof. From (3.3.20) it follows that for any i there exist β_i, γ_i in V such that

$$\omega = \alpha_i \wedge \beta_i + \alpha_{k+1} \wedge \gamma_i$$

Therefore for any $1 \leq i \neq j \leq k$

$$\alpha_i \wedge \beta_i - \alpha_j \wedge \beta_j + \alpha_{k+1} \wedge (\gamma_i - \gamma_j) = 0.$$

Taking into account that $\alpha_i, \alpha_j, \alpha_{k+1}$ are linearly independent, we can apply the classical Cartan lemma to conclude that

$$\beta_i \in \operatorname{span}\{\alpha_i, \alpha_j, \alpha_{k+1}\}. \tag{3.3.22}$$

If $k > 2$, then for a given $i \in \{1, \dots, k\}$ we can choose two different elements $\{1, \dots, k\} - \{i\}$ as j in (3.3.22), which implies that

$$\beta_i \in \operatorname{span}\{\alpha_i, \alpha_{k+1}\} \tag{3.3.23}$$

and therefore (3.3.21) holds.

If $k = 2$ then again by Cartan's lemma

$$\beta_1 = c\alpha_2 \mod (\alpha_1, \alpha_3), \quad \beta_2 = -c\alpha_1 \mod (\alpha_1, \alpha_3),$$

for some constant c, which implies (3.3.21) for this case. □

Now let us prove that if $\pi_{\mathcal{R}} \circ \mathrm{Pl}$ is strongly non-injective, then $\dim L$ must be equal to 6 and $\mathcal{R}$ is self-adjoint. Since $\dim L \geq 4$ by Lemma 3.3.7 , without loss of generality, assume that $\{\alpha_i\}_{i=1}^4$ are linearly independent. Since for any λ_1, λ_2 we have $\lambda_1\alpha_1 \wedge \sigma_1 + \lambda_2\alpha_2 \wedge \sigma_2 \in O_5$ and also α_1, α_2 are linearly independent, we are in the Case 1 of Lemma 3.3.5 and by Remark 3.3.6. Hence there exist $\sigma \in \wedge^2 V$ such that (3.3.19) holds. Therefore,

$$\lambda_1\alpha_1 \wedge \sigma_1 + \lambda_2\alpha_2 \wedge \sigma_2 = (\lambda_1\alpha_1 + \lambda_2\alpha_2) \wedge \sigma.$$

Then

$$\mu_1(\lambda_1\alpha_1 \wedge \sigma_1 + \lambda_2\alpha_2 \wedge \sigma_2) + \mu_2\lambda_3\alpha_3 \wedge \sigma_3 = \mu_1(\lambda_1\alpha_1 + \lambda_2\alpha_2) \wedge \sigma + \mu_2\lambda_3\alpha_3 \wedge \sigma_3 \in O_5.$$

Using Lemma 3.3.5 again we get that

$$\sigma = \sigma_3 \mod \Big((\lambda_1\alpha_1 + \lambda_2\alpha_2), \alpha_3\Big), \quad \forall \lambda_1, \lambda_2. \tag{3.3.24}$$

Plugging (λ_1, λ_2) from the set $\{(1,0),(0,1)\}$ into (3.3.24) we get the setting of Lemma 3.3.8 for $\omega = \sigma - \sigma_3$ and $k = 2$. Therefore,

$$\sigma - \sigma_3 = c\alpha_1 \wedge \alpha_2 \mod \alpha_3$$

for some constant c . Hence there exists $\beta \in V$ such that if we set

$$\sigma - c\alpha_1 \wedge \alpha_2 = \sigma_3 + \alpha_3 \wedge \beta.$$

Set $\tilde{\sigma} := \sigma - c\alpha_1 \wedge \alpha_2$. Then

$$\alpha_1 \wedge \sigma_1 = \alpha_1 \wedge \tilde{\sigma}, \quad \alpha_2 \wedge \sigma_2 = \alpha_2 \wedge \tilde{\sigma}, \quad \alpha_3 \wedge \sigma_2 = \alpha_3 \wedge \tilde{\sigma}. \tag{3.3.25}$$

Further, we have that

$$\mu_1(\lambda_1\alpha_1 \wedge \sigma_1 + \lambda_2\alpha_2 \wedge \sigma_2 + \lambda_3\alpha_3 \wedge \sigma_3) + \mu_2\alpha_4 \wedge \sigma_4$$
$$= \mu_1(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3) \wedge \tilde{\sigma} + \mu_2\alpha_4 \wedge \sigma_4 \in O_5$$

This together with Lemma 3.3.5 implies that

$$\tilde{\sigma} = \sigma_4 \mod ((\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3), \alpha_4), \quad \forall \lambda_1, \lambda_2, \lambda_3. \tag{3.3.26}$$

Plugging $(\lambda_1, \lambda_2, \lambda_3)$ from the set $\{(1,0,0), (0,1,0), (0,0,1)\}$ into (3.3.26) we get the setting of Lemma 3.3.8 for $\omega = \tilde{\sigma} - \sigma_4$ and $k = 3$. Therefore

$$\tilde{\sigma} = \sigma_4 \mod \alpha_4,$$

which implies that in addition to (3.3.19) we have $\alpha_4 \wedge \sigma_4 = \alpha_4 \wedge \tilde{\sigma}$.

Further, for α_5 there must exist $i, j, k \in \{1,2,3,4\}$ such that $\alpha_i, \alpha_j, \alpha_k, \alpha_5$ are linearly independent, otherwise $\alpha_5 = 0$, which is impossible. By the same argument as above (with α_4 replaced by α_5, we must have $\alpha_5 \wedge \tilde{\sigma} = \alpha_5 \wedge \sigma_5$. The same argument holds for α_6 as well.

We proved that $\tilde{\sigma} = \sigma_i \mod \alpha_i$ for all $1 \leq i \leq 6$. Since $\alpha_i \wedge \sigma_i = \alpha_i \wedge \tilde{\sigma}$ form a basis of the 6-dimensional $\mathcal{R}$, we must have $\{\alpha_i\}_{i=1}^6$ linearly independent. So we must have $\dim L = 6$ and $\mathcal{R} = span\{\alpha_i \wedge \tilde{\sigma}\}_{i=1}^6$, i.e. $\mathcal{R}$ is self-adjoint.

4. APPLICATIONS TO WRONSKI MAP ON SOLUTION SPACES

In this section we show that the Wronski map $\mathrm{Wr}_{V,m}$ on the Grassmannian $\mathrm{Gr}_m(V)$ of an n-dimensional vector space V of functions is equivalent to the map $\pi_{\mathcal{R}_V} \circ \mathrm{Pl}$ for some subspace $\mathcal{R}_V$ of $\wedge^m V$ and that V is the space of solutions of a homogeneous equation corresponding to a self-adjoint differential operator if and only if the corresponding defining subspace $\mathcal{R}_V$ is self-adjoint in the sense of Definition 1.5.1. This allows us to apply the results of sections 2 and 3 to the Wronski map $\mathrm{Wr}_{V,m}$.

4.1 Curves of subspaces associated with spaces of functions

Definition 4.1.1. Given an n-dimensional vector space V of function on an interval $I \subset \mathcal{R}$ let

$$C_V(t) := \{p \in V^* : \langle p, x(\cdot)\rangle = 0 \quad \forall\, x(\cdot) \in V \text{ such that } \quad x(t) = 0\}, \forall t \in I, \quad (4.1.1)$$

where by $\langle p, x(\cdot)\rangle$ we mean the value of the functional p on the vector $x(\cdot)$. The curve of subspace $C_V(t)$ of V^* is called the *curve associate with the space* V.

Directly from the definition $\dim C_V(t) \leq 1$. Take a smooth curve $c(t)$ in V^*, satisfying $c(t) \in C_V(t)$ and $\mathrm{span}\{c(t)\} = C_V(t)$ In this case we say that $c(t)$ is a section of the curve $C_V(t)$. Let

$$C_V^{(i)}(t) = \mathrm{span}\left(c(t), c'(t), \ldots, c^{(i)}(t)\right). \quad (4.1.2)$$

Obviously, the subspace $C_V^{(i)}(t)$ does not depend on the choice of the section $c(t)$. It is called the *ith osculating (or the ith tangent developable) space of the curve*

$\tau \mapsto C_V(\tau)$ *at the point* t.

It is clear that $\dim C_V^{(i)}(t) - \dim C_V^{(i-1)}(t) \leq 1$.

Proposition 4.1.2. *If* $\big(\phi_1(\cdot), \dots \phi_n(\cdot)\big)$ *is a basis in the space* V *of solutions of the equation* $Lx = 0$ *and* $(e_1, \dots, e_n)$ *is the basis in* V^* *dual to this basis, then*

$$C_V(t) = \operatorname{span}\left\{\sum_{j=1}^{n} \phi_j(t) e_j\right\} \tag{4.1.3}$$

Proof. Take any $x(\cdot) \in V$ such that $x(t) = 0$. Then $x(\cdot) = \sum_{j=1}^{n} \alpha_j \phi_j(\cdot)$ such that

$$\sum_{j=1}^{n} \alpha_j \phi_j(t) = 0.$$

Hence

$$\langle \sum_{j=1}^{n} \phi_j(t) e_j, x(\cdot) \rangle = \langle \sum_{j=1}^{n} \phi_j(t) e_j, \sum_{j=1}^{n} \alpha_j \phi_j(\cdot) \rangle = \sum_{j=1}^{n+1} \alpha_j \phi_j(t) = 0,$$

which implies (4.1.3). □

The last proposition has a direct consequence in the case when the functional space V satisfies the condition (1.2.4), i.e., according to Proposition 1.2.5, V is the space of solution of certain homogeneous linear differential equation of order n. Namely, we have

Corollary 4.1.3. *If the functional space* V *satisfies the condition* (1.2.4), *then*

$$C_V^{(n-1)}(t) = V^* \tag{4.1.4}$$

for any t *or, equivalently,* $\dim C_V^{(i)}(t) = i + 1$ *for all* i *such that* $1 \leq i \leq n$. *In this case one says that the curve* $C_V(t)$ *is a regular (or convex) curve in the projective* *space* $\mathbb{P}V^*$.

Proof. The regularity of $C_V(t)$ follows from the fact that the tuple of vectors

$$\{(\phi_1^{(j)}(t), \dots, \phi_n^{(j)}(t))\}_{j=0}^n$$

is linear independent for any t for a fundamental system of solutions.

□

Note that for the trivial operator $L_0 x = x^{(n)}$ of order n we can take

$$\big(\phi_1(t), \dots, \phi_n(t)\big) = (1, t, \dots, t^{n-1})$$

as a fundamental set of solutions. By the previous lemma the curve C_{L_0} is nothing but the rational normal curve in $\mathbb{P}V$.

As a consequence of Proposition 4.1.2 it follows that if Ω is a volume form on V^* and $c(t)$ is a section of the curve $C_V(t)$ satisfying

$$c(t) = \sum_{j=1}^{n} \phi_j(t) e_j, \tag{4.1.5}$$

where $\big(\phi_1(\cdot), \dots \phi_n(\cdot)\big)$ and $(e_1, \dots, e_n)$ are as in Proposition 4.1.2, then, up to a constant multiple,

$$\mathrm{Wr}(V) = \Omega\big(c(t), c'(t), \dots, c^{(n-1)}(t)\big). \tag{4.1.6}$$

Remark 4.1.4. Note that if V satisfies (1.2.4) then for any section $c(t)$ of the curve $C_V(t)$ the vectors $c(t), c'(t), \dots, c^{(n-1)}(t)$ constitute a basis of the space V^* for any t. It is easy to show that there exist a unique, up to a constant scalar multiple, section

$c(t)$ of C such that

$$c^{(n)}(t) \in \mathrm{span}\{\mathrm{c(t), c'(t), \ldots, c^{(n-2)}(t)}\}. \tag{4.1.7}$$

A section satisfying the last property is called the *canonical section of the curve* $C_V(t)$. $\square$

4.2 The defining subspace for the Wronski map of the space of functions

Let as before V be an n-dimensional vector space of functions on an open interval $I \subset \mathbb{R}$. Let Λ be a subspace of dimension m in V. Consider the curve $C_\Lambda(t)$ of subspaces in $\mathbb{P}\Lambda^*$. Note that the space Λ^* can be identified canonically with the space $V^*/\Lambda^\perp$, where $\Lambda^\perp$ be as in (3.1.3). For this we assign to $p \in \Lambda^*$, which is a linear functional on Λ, the set of all its linear extensions to V. Since the difference between any two such extension vanishes on Λ, this set is nothing but an element of $V^*/\Lambda^\perp$. Under thi identification the canonical projection $\pi : V^* \to V^*/\Lambda^\perp$ sends a linear form on V to its restriction to Λ,

Lemma 4.2.1. *The curve $C_\Lambda(t)$ is the image of the curve $C_V(t)$ under the canonical quotient map $\pi : V^* \to V^*/\Lambda^\perp$.*

Proof. Indeed, from (4.1.1) it follows that if $p \in C_V(t)$, then $\langle p, x(\cdot)\rangle = 0$ for all $x(\cdot) \in V$ such that $x(t) = 0$. Since $\pi(p)$ is the restriction of p to Λ we get that $\pi(p) \in C_\Lambda(t)$.

Vice versa, if $\hat{p} \in C_\Lambda(t)$, then $\langle p, x(\cdot)\rangle = 0$ for all $x(\cdot) \in \Lambda$ such that $x(t) = 0$ and it can be extended to $p \in V^*$ such that $\langle p, x(\cdot)\rangle = 0$ for all $x(\cdot) \in V$ such that $x(t) = 0$, so $\hat{\pi} = \pi(p)$. $\square$

Now assume as before that $\big(\phi_1(\cdot), \ldots \phi_n(\cdot)\big)$ is a basis in the space V, $(e_1, \ldots, e_n)$

is the basis in V^* dual to this basis, and $c(t) = \sum_{j=1}^{n} \phi_j(t) e_j$. Assume that $\widetilde{\Omega}$ is a volume form on the space V^* and $(a_1, \ldots, a_{n-m})$ is a basis of the space $\Lambda^\perp$. Then the volume form $\widetilde{\Omega}$ induces the volume form Ω on $V^*/\Lambda^\perp$ ($\cong \Lambda^*$) as follows

$$\Omega\left(\pi(v_1), \ldots, \pi(v_m)\right) = \widetilde{\Omega}(v_1, \ldots, v_m, a_1, \ldots, a_{n-m}).$$

Then combining Lemma 4.2.1 and formula (4.1.6) (applied to Λ instead of V) we get that, up to a scalar multiple,

$$\mathrm{Wr}(\Lambda) = \widetilde{\Omega}\big(c(t), c'(t), \ldots, c^{(m-1)}(t), a_1, \ldots, a_{n-m}\big). \tag{4.2.1}$$

Note that similar formulas were used by many authors (for example, [3]). Now define the following subspace of V^*

$$K_V := \mathrm{span}_{t \in I}\{\mathrm{Pl}\big(C^{(m-1)}(t)\big)\} \subset \wedge^m V^*\} \tag{4.2.2}$$

where by $\mathrm{Pl}\big(C^{(m-1)}(t)\big)$ we mean the line in $\wedge^m V^*$ corresponding to the point $\mathrm{Pl}\big(C^{(m-1)}(t)\big)$ of $\mathbb{P} \wedge^m V^*$ if $\dim C^{m-1}(t) = m$ and we set $\mathrm{Pl}\big(C^{(m-1)}(t)\big) = 0$ if $\dim C^{m-1}(t) < m$. Now assume that

$$\mathcal{R}_V := (K_V)^\perp = \{\omega \in (\wedge^m V^*)^* : \omega|_{K_V} = 0\} \tag{4.2.3}$$

Since the dual space $(\wedge^m V^*)^*$ of $\wedge^m V^*$ is naturally identified with $\wedge^m V$, the space $\mathcal{R}_V$ is in fact a subspace of $\wedge^m V$.

Proposition 4.2.2. *The maps* $\mathrm{Wr}_{V,m}$ *and* $\pi_{\mathcal{R}_V} \circ \mathrm{Pl}$*, considered as maps from* $\mathrm{Gr}_m(V)$*, are equivalent in the sense of Definition 1.3.1. In particular, they have the same injectivity properties.*

Proof. Fix a basis of V as $E = \{e_1, e_2, \ldots, e_n\}$ in V and let $\{e_1^*, e_2^*, \ldots, e_n^*\}$ be the dual basis in V^*, $e_i^*(e_j) = \delta_{i,j}$. By sending e_i to e_i^* we identify V and V^*.

Given $\Lambda \in \mathrm{Gr}_m(V)$ denote by $\Lambda_E^\perp$ the element of $\mathrm{Gr}_{n-m}(V)$ corresponding to the subspace $\Lambda^\perp \in V^*$ under the above identification. Let $(a_1, \ldots, a_{n-m})$ be a basis of the space $\Lambda_E^\perp$.Let A be a $(n-m) \times n$ matrix with *ith* row being equal to the coordinates of the vector a_i with respect to the basis $\{e_1, e_2, \ldots, e_n\}$. Let Ind_s be the set of strongly monotone multiindices of length s with entries belonging to $\{1, \ldots, n\}$: $I \in \mathrm{Ind}_s$ if $I = (i_1, \ldots, i_s)$ with integer i_k such that $1 \leq i_1 < i_2 < \ldots < i_s \leq n$. Given $I \in \mathrm{Ind}_{n-m}$ let A_I be the $(n-m)\times(n-m)$ minors of the matrix A with columns from I. Also, given a multiindex $I = (i_1, \ldots, i_s) \in \mathrm{Ind}_s$ let $E_I = e_{i_1} \wedge \ldots \wedge e_{i_s}$. Further, given a multiindex $I \in \mathrm{Ind}_m$ let $\bar{I}$ be multiindex from Ind_{n-m} which complete I to $\{1, \ldots, n\}$. Then

$$\mathrm{Pl}(\Lambda_E^\perp) = \Big[\sum_{I \in \mathrm{Ind}_{n-m}} A_I E_I\Big] = \Big[\sum_{I \in \mathrm{Ind}_m} A_{\bar{I}} E_{\bar{I}}\Big]. \tag{4.2.4}$$

Lemma 4.2.3. *The following identity hold*

$$\mathrm{Pl}(\Lambda) = \Big[\sum_{I \in \mathrm{Ind}_m} (-1)^{|I|} A_{\bar{I}} E_I\Big]. \tag{4.2.5}$$

Proof. The basis E in V defines an inner product on V. Let $* : \wedge^{n-m} V \to \wedge^m V$ be the corresponding Hodge star operator. We also consider this operator as a map of the corresponding projective spaces. Then directly from definition of the Hodge star and the fact that the space $\Lambda_E^\perp$ is orthogonal to Λ in V it follows that

$$\mathrm{Pl}(\Lambda) = *\mathrm{Pl}(\Lambda_E^\perp) \tag{4.2.6}$$

On the other hand, from linearity of the Hodge star operator it follows that

$$\mathrm{Pl}(\Lambda) = *\mathrm{Pl}(\Lambda_E^{\perp}) = \Big[*\Big(\sum_{I\in \mathrm{Ind}_m} A_{\bar I}E_{\bar I}\Big)\Big] = \Big[\sum_{I\in \mathrm{Ind}_m} A_{\bar I}(*E_{\bar I})\Big]. \tag{4.2.7}$$

Then it is easy ti see that

$$*(E_{\bar I}) = (-1)^{|I|-m}E_I. \tag{4.2.8}$$

Substituting (4.2.8) to (4.2.7) we get (4.2.5).

□

Now let $c(t)$ be a section of the curve $C(t)$ and $M(t)$ be the $m \times m$ matrix with the ith row being the coordinates of the vector $c^{(i)}(t)$ with respect to the basis $\{e_1^*, e_2^*, \ldots, e_n^*\}$. For any multiindex $I \in \mathrm{Ind}_m$ let $M_I(t)$ be the $m \times m$ minor of the matrix $M(t)$ with columns from I. Then by (4.1.6)

$$\mathrm{Wr}_{V,m}(\Lambda)(t) = \big[\sum_{I\in \mathrm{Ind}_m}(-1)^{|I|}A_{\bar I}M_I(t)\big]. \tag{4.2.9}$$

Now define the linear map $\widehat\Psi$ from $\wedge^m V$ to the linear span of the tuple of functions $\{M_I(\cdot)\}_{I\in \mathrm{Ind}_m}$ by setting $\widehat\Psi(E_I) := M_I(\cdot)$. From relations (4.2.5) and (4.2.9) it follows that

$$\widehat\Psi\big(\mathrm{Pl}(\Lambda)\big) = \mathrm{Wr}_{V,m}(\Lambda). \tag{4.2.10}$$

(in the last equation we look on Ψ as on the map on $\mathbb{P}\wedge^m V$).

Note that directly from definition given (4.2.3) the space $\mathcal{R}_V$ satisfies

$$\mathcal{R}_V = \ker\,\widehat\Psi. \tag{4.2.11}$$

Therefore there exists a linear isomorphism Ψ between $\wedge^m V/\mathcal{R}_V$ and the linear span of the tuple of functions $\{M_I(\cdot)\}_{I\in \mathrm{Ind}_m}$ such that

$$\widehat{\Psi} = \Psi \circ \hat{\pi}_{\mathcal{R}_V}, \tag{4.2.12}$$

where $\hat{\pi}_{\mathcal{R}_v} : \wedge^m V \to \wedge^m V/\mathcal{R}_V$ is the canonical projection. The isomorphism Ψ defines a bijection $\widetilde{\Psi}$ between $\mathbb{P}\wedge^m V$ and $\mathbb{P}\wedge^m V/\mathcal{R}_V \cup \{0\}$. Therefore, combining this with (4.2.10) and (4.2.12) we will get that

$$\widetilde{\Psi} \circ \big(\pi_{\mathcal{R}_V} \circ \mathrm{Pl}\big)(\Lambda) = \mathrm{Wr}_{V,m}(\Lambda), \quad \forall \Lambda \in \mathrm{Gr}_m(V).$$

So, the maps $\mathrm{Wr}_{V,m}$ and $\pi_{\mathcal{R}_V} \circ \mathrm{Pl}$ are equivalent. □

4.3 Symplectic form on the space of solutions of self-adjoint operators

In this subsection we will work with the space V_L of solutions of homogeneous linear differential equation corresponding to the linear differential operator L of order n as in (1.2.1) and we will show that L is self-adjoint in the classical sense if and only the defining subspace $\mathcal{R}_{V_L}$ is self adjoint in the sense of Definition 1.5.1. The presentation in this section mostly uses results from [9]. Some of the construction can be found already in [14].

We start with the following definition:

Definition 4.3.1. Two curves $C(t)$ and $\widetilde{C}(t)$ in the projective spaces $\mathbb{P}V$ and $\mathbb{P}\widetilde{V}$ of linear spaces V and $\tilde{V}$ are called *equivalent*, if there exists a nonsingular linear isomorphism $A : V \mapsto \widetilde{V}$ sending one curve to another, i.e.

$$A\big(C(t)\big) = \widetilde{C}(t) \quad \forall t. \tag{4.3.1}$$

In this case we will also write that $C \sim \widetilde{C}$.

Remark 4.3.2. If two regular curves $C(t)$ and $\widetilde{C}(t)$ in projective spaces are equivalent, then a linear isomorphism A sending one curve to another as in (4.3.1) is defined up to a scalar multiple. It follows from the following three facts based on the properties of the canonical sections of a curve in projective space defined in Remark4.1.4: firstly, A must send a canonical section of C to a canonical section of $\widetilde{C}$, secondly, canonical sections are defined by a nonzero constant scalar multiple, and finally by the regularity (convexity) assumption, their derivatives up to the order n span the corresponding vector spaces. $\square$.

For shortness denote the curve in projective space $\mathbb{P}V_L$ associated with the space of function V_L by C_L and we say that the curve C_L is *associated with the operator* L.

Remark 4.3.3. Note that two curves C_L and $C_{\widetilde{L}}$, associated with the linear operators L and $\widetilde{L}$, are equivalent if and only if operators L and $\widetilde{L}$ are equivalent in the sense of Definition 1.2.2 The reason again is that a section of a curve is sent to a section of a curve by an equivalence map.

Further, for any regular curve C in $n-1$-dimensional projective space $\mathbb{P}W$ of n-dimensional vector space W set

$$C^*(t) = \left(C^{(n-2)}(t)\right)^{\perp} = \left\{p \in W^* : p(x) = 0, \forall\, x \in C^{(n-2)}(t)\right\} \tag{4.3.2}$$

By the regularity dim $C^{(n-2)}(t) = n-1$. Therefore dim $C^*(t) = 1$. In other words, $t \mapsto C^*(t)$ is a curve in $\mathbb{P}W^*$. This curve is called the *dual curve* to the curve C.

Definition 4.3.4. The curve C in $\mathbb{P}$ is called self-dual if it is equivalent to its dual curve C^* in the sense of Definition refequivcurve.

The following proposition gives a link between the operations of taking the adjoint of differential operators and taking the dual of the corresponding curves in projective spaces.

Proposition 4.3.5. *(see [9]) The curve $(C_L)^*$, which is dual to the curve $C_L(t)$ corresponding to the operator L, is equivalent to the curve $C_{(-1)^n L^*}(t)$ associated with the adjoint operator L^* or shortly*

$$(C_L)^*(t) \sim C_{(-1)^n L^*}(t). \tag{4.3.3}$$

As a direct consequence of it we get the following

Corollary 4.3.6. *The linear differential operator L of order n is equivalent to self-adjoint for even n or anti-self-adjoint for odd n , i.e. $L^* = (-1)^n L$, if and only if the corresponding curve C_L is self-dual.*

Now assume that C is self-dual in $\mathbb{P}W$. Then there exists a unique , up to a non-zero scalar multiple, linear isomorphism $A : W \mapsto W^*$ such that $A\big(C(t)\big)= C^*(t)$ holds for any t. Then we can define the following bilinear forms σ and σ^* on W and on W^*, respectively:

$$\begin{aligned} &\sigma(u, v) = \langle Au, v\rangle, \quad u, v \in W,\\ &\sigma^*(u^*, v^*) = \langle u^*, A^{-1}v^*\rangle, \quad u^*, v^* \in W^*. \end{aligned} \tag{4.3.4}$$

Since A is an isomorphism these forms are nondegenerate.

Proposition 4.3.7. *([9]) The bilinear forms σ, associated with a self-dual curve C is antisymmetric if $\dim W$ is even and symmetric if $\dim W$ is odd.*

If $n = 2m$ and $L^* = L$, then by Corollary refselfdualcor the curve C_L is self-dual. Let σ_L and σ_L^* be the corresponding bilinear forms on V_L and V_L^* defined up to multiplication by a constant. Then these forms are skew-symmetric and nondegenerate, i.e. they define the (conformal) symplectic structures on V_L and V_L^*.

Definition 4.3.8. The forms σ_L and σ_L^* are called the *canonical (conformal) symplectic forms* on the spaces V_L and V_L^*.

Recall that an m-dimensional subspace Λ of a $2m$-dimensional space W endowed with a symplectic form σ is called *Lagrangian* (with respect to σ) if $\sigma|_\Lambda = 0$, i.e., $\sigma(u, v) = 0$ for any $u, v \in \Lambda$.

Proposition 4.3.9. *([9]) If a linear differential operator L is equivalent to a self-adjoint operator of order $2m$, $C_L(t)$ is the curve in $\mathbb{P}V_L$ associated to it, and σ_L^* is the canonical symplectic form on V_L^*, then for any $t \in I$ the $(m-1)$th osculating subspace $C_L^{(m-1)}(t)$ is a Lagrangian subspace of V_L^* with respect to σ_L^*. Conversely, if for a linear differential operator L of order $2m$ there exists a symplectic form σ^* on V_L^* such that the subspace $C_L^{(m-1)}(t)$ is a Lagrangian subspace of V_L^* with respect to σ^*, then L is equivalent to a self-adjoint operator and σ^* is the canonical symplectic form on V_L^*.*

Proposition 4.3.10. *A linear differential operator L of order $2m$ is equivalent to a self-adjoint operator if and only if the corresponding defining subspace $\mathcal{R}_{V_L}$ in $\wedge^m V_L$ is self-adjoint in the sense of Definition 1.5.1.*

Proof. If L is self-adjoint then from Proposition refLagr and the definition of $\mathcal{R}_{V_L}$ given by (4.2.3) it follows that $\sigma_L^* \wedge \alpha \in \mathcal{R}_{V_L}$ for any $\alpha \in \wedge^{m-2} V_L^*$. Conversely, if $\mathcal{R}_{V_L}$ is self-adjoint , then there exist a nondegenerate $\sigma^* \in \wedge^2 V$ such that $\sigma^* \wedge \alpha \in \mathcal{R}_{V_L}$ for any $\alpha \in \wedge^{m-2} V_L^*$. This means that $\sigma^* \wedge \alpha|_{C^{(m-1)}(t)} = 0$ for any $\alpha \in \wedge^{m-2} V_L^*$ for any $t \in I$. The latter implies that $\sigma^*_{C^{(m-1)}(t)} = 0$ for any $t \in I$, i.e. $C^{(m-1)}(t)$ is Lagrangian with respect to σ^* for any $t \in I$. Hence, by the previous proposition L is equivalent to a self-adjoint operator. $\square$

Proposition 4.3.10 implies that Theorem 1.2.4 is a direct consequence of Theorem 1.5.2. Also as a direct consequence of Remark 1.1.1,Proposition 4.3.10, Corollary 3.3.2 and Theorem 3.3.4 we have

Corollary 4.3.11. *Among all linear differential operators of L order 6 with analytic coefficients and such that the dimension of the corresponding defining subspace $\mathcal{R}_{V_L}$ is not greater than 6, the operators which are equivalent to a self-adjoint operator are the only ones with the Wronski map being strongly noninjective.*

Remark 4.3.12. Note that for a trivial differential operator $L_0 x = x^{(6)}$ of order 6, the dimension of the corresponding defining subspace $\mathcal{R}_{V_{L_0}} = 10$, while it can be shown that for any differential operator L of order 6 different from L_0 we have $\dim \mathcal{R}_{V_L} < 10$. This together with Theorem 3.3.1 and the fact that the trivial operator L_0 is self-adjoint will imply that if for a linear operator of order 6 the defining subspaces $\mathcal{R}_{V_L}$ satisfies $\dim \mathbb{P}\mathcal{R}_{V_L} \cap O_5 \leq 4$ then the corresponding Wronski map $\mathrm{Wr}_{V_L,m}$ is essentially injective.

4.4 The generalized Wronski map associate with a curve in $\mathrm{Gr}_m(V^*)$

Note that the definition of the defining space $\mathcal{R}_V$ for a space of functions V given by (4.2.2)-(4.2.3) depends on the curve of subspaces $C^{(m-1)}(t)$ in V^* and in the case when V satisfies (1.2.4) (i.e. the space of solution of a linear homogeneous differential equation), then this curve is the curve in $\mathrm{Gr}_m(V^*)$. More generally, let V be an abstract n dimensional vector space and $\Gamma(t)$, $t \in I \subset \mathbb{R}$ be a curve in $\mathrm{Gr}_m(V^*)$ which does not necessarily come from osculating a curve in projective space an appropriate number of times. This situation appear for example for linear control systems, where such curve can be constructed from a transfer function (see section 5). Then by analogy with (4.2.2)-(4.2.3) define

$$K^{\Gamma} := \mathrm{span}_{t\in I}\{\mathrm{Pl}\big(\Gamma(t)\big)\} \subset \wedge^m V^*\}, \tag{4.4.1}$$

where by $\mathrm{Pl}\big(\Gamma(t)\big)$ we mean the line in $\wedge^m V^*$ corresponding to the point $\mathrm{Pl}\big(\Gamma(t)\big)$ of $\mathbb{P}\wedge^m V^*$. Now assume that

$$\mathcal{R}^{\Gamma} := (K^{\Gamma})^{\perp} = \{\omega \in \wedge^m V : \omega|_{K^{\Gamma}} = 0\} \tag{4.4.2}$$

Then by complete analogy with Proposition 4.3.10 one can prove the following

Proposition 4.4.1. *The space R^{Γ} is self-adjoint in the sense of Definition 1.5.1 if and only if there exists a symplectic form σ^* in V^* such that the curve $\Gamma(t)$ is the curve of Lagrangian subspaces with respect to σ^*.*

Based on the last Proposition the consequences of Theorems 1.5.2, 3.3.1, 3.3.4, and Corollary 3.3.2 for the generalized Wronski map $\pi_{\mathcal{R}^{\Gamma}} \circ \mathrm{Pl}$ can be formulated appropriately and are left to the reader.

4.5 Analytic description of the canonical symplectic form

Let us finish this section with more analytic description of the canonical symplectic form on the space of solutions of self-adjoint operators. The original motivation for the definition of the adjoint differential operator comes from the following identity obtained by the application of integration by parts $2m$ times: for any a and b

$$\int_a^b Lu\, v\, dt = \int_a^b u\, L^* v\, dt + A_{a,b}(u,v) \tag{4.5.1}$$

where

$$A_{a,b}(u,v) = \sigma_b(u,v) - \sigma_a(u,v) \tag{4.5.2}$$

with

$$\sigma_t(u,v) = \sum_{i=0}^{n+1}\sum_{k=0}^{i-1}(-1)^k(a_iv)^{(k)}(t)u^{(i-k-1)}(t), \quad a_{n+1}(t) \equiv 1$$

.

Proposition 4.5.1. *The form σ_t is skew symmetric and its restriction to the space of solutions $V_L(\mathbb{C})$ of the homogeneous differential equation $Lx = 0$ does not depend on t.*

Proof. If the operator L is self-adjoint, then the bilinear form $A_{a,b}(u,v)$ is skew-symmetric. Since the bilinear form $\sigma_t(u,v)$ depends on the nth jet of u and v only and we can take functions u and v with arbitrary prescribed nth jet at $t = b$ and zero nth jet at $t = a$, then the forms $\sigma_t(u,v)$ are skew-symmetric for any t. Now if we restrict our forms to the space of solutions $V_L(\mathbb{C})$ of the homogeneous equation $Lx = 0$, then since $L^* = L$ from (4.5.1) it follows that the bilinear form $A_{a,b}(u,v)$ vanishes on $V_L(\mathbb{C})$. Then by (4.5.2) the skew-symmetric bilinear form $\sigma_t(u,v)$ is independent of t on $V_L(\mathbb{C})$. This is exactly the symplectic form on the space of solutions $V_L(\mathbb{C})$ we are looking for. □

It can be shown that the form σ_L from the definition (4.3.8) coincides, up to a nonzero scalar multiple, with the form σ_0.

Remark 4.5.2. If $Lx = x^{(2m)}$, then the space of solutions is $Pol_{2m-1}(\mathbb{C})$, the operator L is self-adjoint and the corresponding symplectic form is given by

$$\sigma_0\Big(\sum_{k=0}^{2m-1} a_kt^k, \sum_{k=0}^{2m-1} b_kt^k\Big) = \sum_{k=0}^{2m-1}(-1)^k k!(2m-1-k)!a_kb_{2m-1-k}.$$

□

5. APPLICATION TO LINEAR CONTROL SYSTEMS

5.1 Transfer function and pole placement map of linear control system

Our theory above can be applied to control theory by static output feedback. Suppose that a triple of real matrices $\Sigma = (A, B, C)$ of sizes $N \times N$, $N \times m$ and $p \times N$ is given. This triple Σ defines a linear system

$$\dot{x} = Ax + Bu, \tag{5.1.1}$$

$$y = Cx \tag{5.1.2}$$

where $x \in \mathbb{C}^N, y \in \mathbb{C}^m, x \in \mathbb{C}^p$. The values of x, u and y at a point $t \in \mathbb{R}$ are interpreted as the state, input and output of our system at the moment t. We assume that this system is both controllable and observable, so that its Mcmillan degree is N(See [7]). Applying the Laplace transform to (5.1.1)-(5.1.2) and assuming that $x(0) = 0$ we get that

$$\hat{y}(s) = C(sI - A)^{-1}B\hat{u}(s), \tag{5.1.3}$$

where $\hat{u}(s)$ and $\hat{y}(s)$ are Laplace transforms of the input function $u(t)$ and the output function $y(t)$. The transfer function function

$$G(s) = C(sI - A)^{-1}B \tag{5.1.4}$$

of the control system given by Σ is a function of (in general, complex) variable s with values in the set of $p \times m$ matrices.

One wishes to control a given system by arranging a feedback, which means sending the output to the input via an $m \times p$ matrix K, called a gain matrix:

$$u = Ky. \tag{5.1.5}$$

Elimination of u and y gives the closed loop system

$$\dot{x} = (A + BKC)x,$$

whose transfer function has poles at the zeros of the polynomial

$$\mathrm{Pol}_\Sigma(K)(s) = \det(sI - A - BKC).$$

The map $K \mapsto \mathrm{Pol}_\Sigma(K) \in Poly^N_{\mathbb{C}}$ is called the *pole placement map* and the problem about pole placement assignment is: given a system Σ, and a set $\{z_1, \dots, z_N\}$ to find a gain matrix K, such that the zeros of $\mathrm{Pol}_\Sigma(K)$ are $\{z_1, \dots, z_N\}$. Thus, for a fixed system Σ, arbitrary pole assignment Pol_Σ is possible if and only if the pole placement map is surjective. When $N \leq mp$ this map is surjective. It clear that if $N > mp$ then the map Pol_Σ cannot be surjective (See [13]). Here we are interested in question whether the pole placement map is strongly noninjective.

Since K defines the linear map from Y to U it defines an element of $\mathrm{Gr}_p(Y \times U)$ being the graph of this linear map. Vice versa, any element of $\mathrm{Gr}_p(Y \times U)$ transversal to the subspace $0 \times U$ is the graph of the linear map from Y to U and therefore defines a feedback of the form (1.4.3). Hence, the map Pol_Σ is well defined on the affine coordinate domain of $\mathrm{Gr}_p(Y \times U)$. In the next two subsections we show that this question is a particular case of the framework of generalized Wronski maps described in subsection 4.4.

5.2 A curve in Grassmannian associate with the transfer function and state-feedback transformation

Consider the curve Γ_Σ in $\mathrm{Gr}_m(Y \times U)$ such that $\Gamma_\Sigma(s)$ is the graph of linear operator from U to Y with the matrix $G(s)$ in the standard basis of $\mathbb{C}^p \times \mathbb{C}^m$. We say that Γ_Σ is the *curve in* $\mathrm{Gr}_m(Y \times U)$ *associated with the linear control system* (5.1.1)-(5.1.2).

Now we introduce a natural group of transformations of linear control systems, the state-feedback transformations and show that the transfer functions of state-feedback linear control system define the same curve in $\mathrm{Gr}_m((Y \times U)$ up to the natural action of the GeneralLinear group. foir this consider the following change of coordinates in the state, input, and output spaces

$$\begin{cases} x = L\widetilde{x} \\ u = Q\widetilde{y} + W\widetilde{u} \\ y = T\widetilde{y} \end{cases} \tag{5.2.1}$$

where L, W, and T are nonsingular matrices of sizes $N \times N$, $m \times m$, and $p \times p$ respectively, and Q is an $m \times p$ matrix, The transformation of the space $X \times U \times Y$ given by (5.2.1) is called a *state-feedback transformation* Substituting (5.2.1) into (5.1.1)-(5.1.2), we obtain a new linear control system in $\widetilde{x}$, $\widetilde{u}$, $\widetilde{y}$:

$$\dot{\widetilde{x}} = \widetilde{A}\widetilde{x} + \widetilde{B}\widetilde{u} \tag{5.2.2}$$

$$\widetilde{y} = \widetilde{C}\widetilde{x}, \tag{5.2.3}$$

given by the triple of matrices $\widetilde{\Sigma} = (\widetilde{A}, \widetilde{B}, \widetilde{C})$, where

$$\widetilde{A} = L^{-1}(A + BQT^{-1}C)L), \quad \widetilde{B} = L^{-1}BW, \quad \widetilde{C} = T^{-1}CL. \tag{5.2.4}$$

We say that two linear control systems are *state-feedback* equivalent, if there is a state-feedback transformation transforming one system to another.

Proposition 5.2.1. *Let* $\Gamma_{\widetilde{\Sigma}}$ *be the curve in* $\mathrm{Gr}_m(Y \times U)$ *associated with the linear control system* (5.2.2)-(5.2.3). *Then the curves* Γ_Σ *and* $\Gamma_{\widetilde{\Sigma}}$ *are equivalent in the sense of Definition* (4.3.1).

Proof. Let $\widetilde{G}(s)$ be the transfer function of the control system (5.2.2)-(5.2.3). Since the transition functions in the affine charts are matrix Möbius transformations the Proposition follows from the following transformation rule between the transition functions $G(s)$ and $\widetilde{G}(s)$:

$$\widetilde{G}(s) = T^{-1}G(s)(W^{-1} - W^{-1}QT^{-1}G(s))^{-1} \tag{5.2.5}$$

This transformation rule can be verified by straightforward computations. $\square$

5.3 The pole placement map as a generalized Wronski map

The goal of this subsection is to prove the following:

Proposition 5.3.1. *Assume that the linear control system* (5.1.1)-(5.1.2) *is controllable and observable. Then the pole placement map* Pol_Σ *of the control system* (5.1.1)-(5.1.2) *can be extended to a map on the whole* $\mathrm{Gr}_p(Y \times U)$, *which is equivalent to generalized Wronski map with the defining subspace* $\mathcal{R}^{\Gamma_\Sigma}$ *in* $\wedge^m\Big((Y \times U)^*\Big)$.

Proof. Use the coprime factorization of the transfer function $G(s)$,

$$G(s) = C(sI - A)^{-1}B = E(s)D(s)^{-1}, \quad \det D(s) = \det(sI - A), \tag{5.3.1}$$

where D and E are polynomial matrix-functions of sizes $m\times m$ and $p\times m$, respectively. The polynomial matrix $\begin{pmatrix} D(s) \\ E(s) \end{pmatrix}$ has the following properties: its full size minors have no common zeros, and exactly one of these minors, det $D(s)$, has degree N while all other minors have strictly smaller degree. Using factorization (5.3.1) and the identity $\det(I - PQ) = \det(I - QP)$, which is true for all rectangular matrices of appropriate dimensions, we write

$$\begin{aligned} \mathrm{Pol}_\Sigma(K)(s) &= \det(sI - A - BKC) = \det(I - BKC(sI - A)^{-1}) \det(sI - A) \\ &= \det(I - KC(sI - A)^{-1}B) \det(sI - A) \\ &= \det(I - KE(s)D(s)^{-1}) \det D(s) = \det(D(s) - KE(s)). \end{aligned}$$

This can be rewritten as

$$\mathrm{Pol}_\Sigma(K)(s) = \begin{vmatrix} D(s) & K \\ E(s) & I \end{vmatrix}. \tag{5.3.2}$$

By constructhion, the span of the first m columns of matrix in (5.3.2) coincides with the space $\Gamma_\Sigma(s)$. So, we get the formula analogous to (4.2.1) and the statement of the Proposition follows from the the same arguments as in the proof of Proposition 4.2.2. $\square$

Definition 5.3.2. Given a control system (5.1.1)-(5.1.2), given by the triple of the matrices $\Sigma = (A, B, C)$, the space $\mathcal{R}^{\Gamma_\Sigma}$ is called the *defining subspaces associated* *with this control system.*

5.4 Symmetric linear control systems

Now we have the following characterization of linear control system such that the defining subspace $\mathcal{R}^{\Gamma_\Sigma}$ is self-adjoint in the sense of Definition (1.5.1).

Definition 5.4.1. A linear control system (5.1.1)-(5.1.2) is called symmetric if $A^T = A$ and $C = B^T$.

Proposition 5.4.2. *For a controllable and observable control system* (5.1.1)-(5.1.2), *given by the triple of the matrices* $\Sigma = (A, B, C)$*, the corresponding defining subspace* $\mathcal{R}^{\Gamma_\Sigma}$ *is self-adjoint in the sense of Definition* (1.5.1) *if and only if this system is state-feedback equivalent to a symmetric one.*

Proof. According to Proposition 4.4.1 the defining subspace $\mathcal{R}^{\Gamma_\Sigma}$ is self-adjoint in the sense if and only if the curve Γ_Σ is a curve of Lagrangian subspaces with respect to some symplectic form on $U \times Y$. Then by Proposition 5.2.1 and the fact that Lagrange Grassmannian is parametrized by symmetric matrices in an appropriate affine chart it follows that $\mathcal{R}^{\Gamma_\Sigma}$ is self-adjoint if and only if the control system (5.1.1)-(5.1.2) is state-feedback equivalent to the linear control system with the transfer function taking values in symmetric matrices. By [4] the latter system is symmetric. $\square$

The strong noninjectivity of symmetric linear control system is clear, because in this case $\mathrm{Pol}_\Sigma(K^T) = \mathrm{Pol}_\Sigma(K)$. The following Corollary is the direct consequence of Corollary 3.3.2 and Theorem 3.3.4

Corollary 5.4.3. *Among all controllable and observable control system with* $m = p = 3$ *with the defining function* $\mathcal{R}^{\Gamma_\Sigma}$ *satisfying:* $\mathcal{R}^{\Gamma_\Sigma} \cap O_{10} = \emptyset$ *and* $\dim \mathcal{R}^{\Gamma_\Sigma} \leq 6$*, the only systems with the strongly noninjective pole placement map are systems, which are state-feedback equivalent to a symmetric one.*

6. SUMMARY

The generalized Wronski map is of great interest since its broad application to other field of math problems. In my thesis, I only answered the case when $m = 2$ and gave a partial answer to the case when $m = 3$. In fact, we can further study the injectivity properties of generalized Wronski map in cases when $m = 4, 5 \ldots$ or even find a general answer to an arbitrary m. But seen from now, the method of using orbits may be more complicated with bigger m, and perhaps other machinery is needed to solve this problem.

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